C^*-pseudo-Kac systems and duality for coactions of concrete Hopf C^*-bimodules

Thomas Timmermann

timmermt@math.uni-muenster.de

SFB 478 "Geometrische Strukturen in der Mathematik"
Hittorfstr. 27, 48149 Münster

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Abstract

We study coactions of concrete Hopf C^*-bimodules in the framework of (weak) C^*-pseudo-Kac systems, define reduced crossed products and dual coactions, and prove an analogue of Baaj-Skandalis duality.

1 Introduction

In a seminal article [1], Baaj and Skandalis developed a duality theory for coactions of Hopf C^*-algebras on C^*-algebras that extends the Takesaki-Takai duality of actions of locally compact abelian groups to all locally compact groups and, more generally, to all regular locally compact quantum groups. More precisely, Baaj and Skandalis introduce the notion of a Kac system which consists of a regular multiplicative unitary and an additional symmetry, consider coactions of the two Hopf C^*-algebras (the "legs") of the multiplicative unitary, define for every coaction of each leg a reduced crossed product that carries a coaction of the other leg, and show that two applications of this construction yield a stabilization of the original coaction.

In this article, we extend the duality theory of Baaj and Skandalis to coactions of concrete Hopf C^*-bimodules, applying the methods and concepts introduced in [10] to the constructions in [1]. In particular, our theory covers coactions of the Hopf C^*-bimodules associated to a locally compact groupoid. An article on examples is in preparation.

Let us mention that a similar duality theory like the one presented here was developed in the PhD thesis of the author [9]. Our new approach allows us to drop a rather restrictive condition (decomposability) needed in [9], and to work in the framework of C^*-algebras instead of the somewhat exotic C^*-families. Moreover, the approach presented here greatly simplifies the complex assumptions needed [9].

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Organization This article is organized as follows:

First, we fix notation and terminology, and summarize some background on (Hilbert) C^*-modules.

In Section 2, we recall the definition of C^*-pseudo-multiplicative unitaries and concrete Hopf C^*-bimodules given in [10].

In Section 3, we introduce weak C^*-pseudo-Kac systems which provide the framework for the construction of reduced crossed products.

In Section 4, we consider coactions of concrete Hopf C^*-bimodules and construct reduced crossed products and duals for such coactions.

In Section 5, we introduce C^*-pseudo-Kac systems and establish an analogue of Baaj-Skandalis duality for coactions of concrete Hopf C^*-bimodules.

In Section 6, we associate to each locally compact groupoid a C^*-pseudo-Kac system.

Preliminaries   Given a subset Y of a normed space X, we denote by $[Y] \subseteq X$ the closed linear span of Y.

Given a Hilbert space H and a subset $X \subseteq \mathcal{L}(H)$, we denote by X' the commutant of X. Given Hilbert spaces H, K, a C^*-subalgebra $A \subseteq \mathcal{L}(H)$, and a $*$-homomorphism $\pi \colon A \to \mathcal{L}(K)$, we put

$$\mathcal{L}^\pi(H,K) := \{T \in \mathcal{L}(H,K) \mid Ta = \pi(a)T \text{ for all } a \in A\};$$

thus, for example, $A' = \mathcal{L}^{\mathrm{id}_A}(H)$.

We shall make extensive use of (right) C^*-modules, also known as Hilbert C^*-modules or Hilbert modules. A standard reference is [4].

All sesquilinear maps like inner products of Hilbert spaces or C^*-modules are assumed to be conjugate-linear in the first component and linear in the second one.

Let A and B be C^*-algebras. Given C^*-modules E and F over B, we denote the space of all adjointable operators $E \to F$ by $\mathcal{L}_B(E,F)$.

Let E and F be C^*-modules over A and B, respectively, and let $\pi \colon A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Then one can form the internal tensor product $E \otimes_\pi F$, which is a C^*-module over B [4, Chapter 4]. This C^*-module is the closed linear span of elements $\eta \otimes_A \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \otimes_\pi \xi | \eta' \otimes_\pi \xi' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\eta \otimes_\pi \xi) b = \eta \otimes_\pi \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. We denote the internal tensor product by "$\otimes$"; thus, for example, $E \otimes_\pi F = E \otimes_\pi F$. If the representation π or both π and A are understood, we write "$\otimes_A$" or "$\otimes$", respectively, instead of "$\otimes_\pi$".

Given E, F and π as above, we define a *flipped internal tensor product* $F_\pi \otimes E$ as follows. We equip the algebraic tensor product $F \odot E$ with the structure maps $\langle \xi \odot \eta | \xi' \odot \eta' \rangle := \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$, $(\xi \odot \eta) b := \xi b \odot \eta$, and by factoring out the null-space of the semi-norm $\zeta \mapsto \|\langle \zeta | \zeta \rangle\|^{1/2}$ and taking completion, we obtain a C^*-B-module $F_\pi \otimes E$. This is the closed linear span of elements $\xi_\pi \otimes \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi_\pi \otimes \eta | \xi'_\pi \otimes \eta' \rangle = \langle \xi | \pi(\langle \eta | \eta' \rangle) \xi' \rangle$ and $(\xi_\pi \otimes \eta) b = \xi b_\pi \otimes \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. As above, we write "$_A\otimes$" or simply "$\otimes$" instead of "$_\pi\otimes$" if the representation π or both π and A are understood, respectively.

Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma \colon F \otimes E \xrightarrow{\cong} E \otimes F$, $\eta \otimes \xi \mapsto \xi \otimes \eta$.

We shall frequently consider the following kind of C^*-modules. Let H and K be Hilbert spaces. We call a subset $\Gamma \subseteq \mathcal{L}(H,K)$ a *concrete C^*-module* if $[\Gamma \Gamma^* \Gamma] = \Gamma$. If Γ is a concrete C^*-module, then evidently Γ^* is a concrete C^*-module as well, the space $B := [\Gamma^* \Gamma]$ is a C^*-algebra, and Γ is a full right C^*-module over B with respect to the inner product given by $\langle \zeta | \zeta' \rangle = \zeta^* \zeta'$ for all $\zeta, \zeta' \in \Gamma$.

2 C^*-pseudo-multiplicative unitaries and concrete Hopf C^*-bimodules

We recall several constructions and definitions from [10] which are fundamental to the duality theory developed in the following sections.

C^*-bases and C^*-factorizations C^*-bases and C^*-factorizations are simple but convenient concepts used in the definition of the C^*-relative tensor product; for details, see [10, Section 2.1].

A C^*-*base* is a triple $(\mathfrak{B}, \mathfrak{H}, \mathfrak{B}^\dagger)$, shortly written $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$, consisting of a Hilbert space $\mathfrak{H}$ and two commuting nondegenerate C^*-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{H})$. We shall mainly be interested in the following example of C^*-bases: If μ is a proper KMS-weight on a C^*-algebra B, then the triple $(\pi_\mu(B), H_\mu, J_\mu \pi_\mu(B) J_\mu)$ is a C^*-base, where H_μ denotes the GNS-space, $\pi_\mu \colon B \to \mathcal{L}(H_\mu)$ the GNS-representation, and $J_\mu \colon H_\mu \to H_\mu$ the modular conjugation associated to μ.

A C^*-*factorization* of a Hilbert space H with respect to a C^*-base $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ is a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{H}, H)$ satisfying $[\alpha^*\alpha] = \mathfrak{B}$, $[\alpha\mathfrak{B}] = \alpha$, and $[\alpha\mathfrak{H}] = H$. We denote the set of all C^*-factorizations of a Hilbert space H with respect to a C^*-base $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ by C^*-fact$(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$.

Let α be a C^*-factorization of a Hilbert space H with respect to a C^*-base $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$. Then α is a concrete C^*-module and a full right C^*-module over $\mathfrak{B}$ with respect to the inner product $\langle \xi | \xi' \rangle := \xi^* \xi'$. We shall frequently identify $\alpha \oslash \mathfrak{H}$ with H via the unitary

$$\alpha \oslash \mathfrak{H} \xrightarrow{\cong} H, \quad \xi \oslash \zeta \mapsto \xi\zeta. \tag{1}$$

There exists a nondegenerate and faithful representation $\rho_\alpha \colon \mathfrak{B}^\dagger \to \mathcal{L}(\alpha \oslash \mathfrak{H}) \cong \mathcal{L}(H)$ such that for all $b^\dagger \in \mathfrak{B}^\dagger$, $\xi \in \alpha$, $\zeta \in \mathfrak{H}$,

$$\rho_\alpha(b^\dagger)(\xi \oslash \zeta) = \xi \oslash b^\dagger \zeta \quad \text{or, equivalently,} \quad \rho_\alpha(b^\dagger)\xi\zeta = \xi b^\dagger \zeta.$$

Let K be a Hilbert space. Then each unitary $U \colon H \to K$ induces a map

$$U_* \colon C^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}) \to C^*\text{-fact}(K; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}), \quad \alpha \mapsto U\alpha,$$

and $\rho_{(U\alpha)}(b^\dagger) = U\rho_\alpha(b^\dagger)U^*$ for all $\alpha \in C^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$ and $b^\dagger \in \mathfrak{B}^\dagger$ because

$$\rho_{(U\alpha)}(b^\dagger)U\xi\zeta = U\xi b^\dagger \zeta = U\rho_\alpha(b^\dagger)\xi\zeta \quad \text{for all } \xi \in \alpha, \, \zeta \in \mathfrak{H}. \tag{2}$$

Let β be a C^*-factorization of K with respect to $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$. We put

$$\mathcal{L}(H_\alpha, K_\beta) := \big\{ T \in \mathcal{L}(H, K) \,\big|\, T\alpha \subseteq \beta, \, T^*\beta \subseteq \alpha \big\}.$$

Evidently, $\mathcal{L}(H_\alpha, K_\beta)^* = \mathcal{L}(K_\beta, H_\alpha)$. Let $T \in \mathcal{L}(H_\alpha, K_\beta)$. Then the map $T_\alpha \colon \alpha \to \beta$ given by $\xi \mapsto T\xi$ is an adjointable operator of C^*-modules, $(T_\alpha)^* = (T^*)_\beta$, and $T\rho_\alpha(b^\dagger) = \rho_\beta(b^\dagger)T$ for all $b^\dagger \in \mathfrak{B}^\dagger$.

Let α be a C^*-factorization of a Hilbert space H with respect to a C^*-base $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ and let $_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}$ be another C^*-base. We call a C^*-factorization $\beta \in C^*\text{-fact}(H; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger})$ *compatible* with α, written $\alpha \perp \beta$, if $\big[\rho_\alpha(\mathfrak{B}^\dagger)\beta\big] = \beta$ and $\big[\rho_\beta(\mathfrak{C}^\dagger)\alpha\big] = \alpha$. In that case, $\rho_\alpha(\mathfrak{B}^\dagger)$ and $\rho_\beta(\mathfrak{C}^\dagger)$ commute. We put $C^*\text{-fact}(H_\alpha; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}) := \{\beta \in C^*\text{-fact}(H; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}) \mid \alpha \perp \beta\}$.

The C^*-relative tensor product The C^*-relative tensor product of Hilbert spaces is a symmetrized version of the internal tensor product of C^*-modules and a C^*-algebraic analogue of the relative tensor product of Hilbert spaces over a von Neumann algebra. We briefly summarize the definition and the main properties; for details, see [10, Section 2.2].

Let H and K be Hilbert spaces, $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ a C^*-base, and $\alpha \in C^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$ and $\beta \in C^*\text{-fact}(K; {}_{\mathfrak{B}^\dagger}\mathfrak{H}_\mathfrak{B})$. The C^*-*relative tensor product* of H and K with respect to α and β is the internal tensor product

$$H \, {}_\alpha\!\!\underset{\mathfrak{H}}{\otimes}_\beta K := \alpha \oslash \mathfrak{H} \oslash \beta.$$

We frequently identify this Hilbert space with $\alpha \oslash_{\rho_\beta} K$ and $H_{\rho_\alpha} \oslash \beta$ via the isomorphisms

$$\alpha \oslash_{\rho_\beta} K \cong H \, {}_\alpha\!\!\underset{\mathfrak{H}}{\otimes}_\beta K \cong H_{\rho_\alpha} \oslash \beta, \quad \xi \oslash \eta\zeta \equiv \xi \oslash \zeta \oslash \eta \equiv \xi\zeta \oslash \eta. \tag{3}$$

3

Using these isomorphisms, we define for each $\xi \in \alpha$ and $\eta \in \beta$ two pairs of adjoint operators

$$|\xi\rangle_{[1]}\colon K \to H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K,\ \zeta \mapsto \xi \ominus \zeta, \qquad \langle\xi|_{[1]} := |\xi\rangle_{[1]}^*\colon \xi' \ominus \zeta \mapsto \rho_\beta(\langle\xi|\xi'\rangle)\zeta,$$

$$|\eta\rangle_{[2]}\colon H \to H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K,\ \zeta \mapsto \zeta \ominus \eta, \qquad \langle\eta|_{[2]} := |\eta\rangle_{[2]}^*\colon \zeta \ominus \eta \mapsto \rho_\alpha(\langle\eta|\eta'\rangle)\zeta.$$

We put $|\alpha\rangle_{[1]} := \big\{|\xi\rangle_{[1]} \,\big|\, \xi \in \alpha\big\}$ and similarly define $\langle\alpha|_{[1]}$, $|\beta\rangle_{[2]}$, $\langle\beta|_{[2]}$.

Let L, M be Hilbert spaces, $\gamma \in \text{C}^*\text{-fact}(L; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$, $\delta \in \text{C}^*\text{-fact}(M; {}_{\mathfrak{B}^\dagger}\mathfrak{H}_\mathfrak{B})$, and $S \in \mathcal{L}(H, L)$, $T \in \mathcal{L}(K, M)$. Using the isomorphisms (3), we define an operator $S \underset{\mathfrak{H}}{\otimes} T \in \mathcal{L}\big(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K,\ L_\gamma \underset{\mathfrak{H}}{\otimes}_\delta M\big)$ in the following cases:

i) if $S \in \mathcal{L}(H_\alpha, L_\gamma)$ and $T\rho_\beta(b) = \rho_\delta(b)T$ for all $b \in \mathfrak{B}$, then $S \underset{\mathfrak{H}}{\otimes} T := S_\alpha \ominus T$;

ii) if $T \in \mathcal{L}(K_\beta, M_\delta)$ and $S\rho_\alpha(b^\dagger) = \rho_\gamma(b^\dagger)S$ for all $b^\dagger \in \mathfrak{B}^\dagger$, then $S \underset{\mathfrak{H}}{\otimes} T := S \ominus T_\beta$.

If $S \in \mathcal{L}(H_\alpha, L_\gamma)$ and $T \in \mathcal{L}(K_\beta, M_\delta)$, then both times $S \underset{\mathfrak{H}}{\otimes} T = S_\alpha \ominus \text{id}_\mathfrak{H} \ominus T_\beta$. Put

$$S_{[1]} := S \underset{\mathfrak{H}}{\otimes} \text{id}\colon H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K \to L_\gamma \underset{\mathfrak{H}}{\otimes}_\beta K, \qquad T_{[2]} := \text{id} \underset{\mathfrak{H}}{\otimes} T\colon H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K \to H_\alpha \underset{\mathfrak{H}}{\otimes}_\delta M.$$

Given C^*-algebras C, D and $*$-homomorphisms $\rho\colon C \to \rho_\alpha(\mathfrak{B}^\dagger)' \subseteq \mathcal{L}(H)$, $\sigma\colon D \to \rho_\beta(\mathfrak{B})' \subseteq \mathcal{L}(K)$, we define $*$-homomorphisms $\rho_{[1]}\colon C \to \mathcal{L}(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K)$, $c \mapsto \rho(c)_{[1]}$, and $\sigma_{[2]}\colon D \to \mathcal{L}(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K)$, $d \mapsto \sigma(d)_{[2]}$.

Let ${}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}$ and ${}_\mathfrak{D}\mathfrak{L}_{\mathfrak{D}^\dagger}$ be C^*-bases. Then there exist compatibility-preserving maps

$$\text{C}^*\text{-fact}(H_\alpha; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}) \to \text{C}^*\text{-fact}(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}), \qquad \gamma \mapsto \gamma \vartriangleleft \beta := \big[|\beta\rangle_{[2]}\gamma\big],$$

$$\text{C}^*\text{-fact}(K_\beta; {}_\mathfrak{D}\mathfrak{L}_{\mathfrak{D}^\dagger}) \to \text{C}^*\text{-fact}(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K; {}_\mathfrak{D}\mathfrak{L}_{\mathfrak{D}^\dagger}), \qquad \delta \mapsto \alpha \vartriangleright \delta := \big[|\alpha\rangle_{[1]}\delta\big],$$

and for all $\gamma \in \text{C}^*\text{-fact}(H_\alpha; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger})$ and $\delta \in \text{C}^*\text{-fact}(K_\beta; {}_\mathfrak{D}\mathfrak{L}_{\mathfrak{D}^\dagger})$,

$$\rho_{(\gamma\vartriangleleft\beta)} = (\rho_\gamma)_{[1]}, \qquad \rho_{(\alpha\vartriangleright\delta)} = (\rho_\delta)_{[2]}, \qquad \gamma \vartriangleleft \beta \perp \alpha \vartriangleright \delta.$$

The C^*-relative tensor product is symmetric, functorial and associative in a natural sense [10, Section 2.2]. Moreover, if the C^*-base ${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ arises from a proper KMS-weight μ on a C^*-algebra B, then $H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K$ can be identified with a von Neumann-algebraic relative tensor product, also known as Connes' fusion; see [10, Section 2.3].

The spatial fiber product of C^*-algebras and concrete Hopf C^*-bimodules

The spatial fiber product of C^*-algebras is a C^*-algebraic analogue of the fiber product of von Neumann algebras [8] and of the relative tensor product of $C_0(X)$-algebras [2]. We briefly summarize the definition and main properties; for details, see [10, Section 3]. Throughout this paragraph, let ${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ be a C^*-base.

A *(nondegenerate) concrete C^*-${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra* (H, A, α) consists of a Hilbert space H, a (nondegenerate) C^*-algebra $A \subseteq \mathcal{L}(H)$, and a C^*-factorization $\alpha \in \text{C}^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$ such that $\rho_\alpha(\mathfrak{B}^\dagger)A \subseteq A$. If (H, A, α) is a nondegenerate concrete C^*-${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra, then $A' \subseteq \rho_\alpha(\mathfrak{B}^\dagger)'$.

Given a Hilbert space H and a C^*-algebra $A \subseteq \mathcal{L}(H)$, we put

$$\text{C}^*\text{-fact}(A; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}) := \{\beta \in \text{C}^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}) \mid (H, A, \beta) \text{ is a concrete } C^*\text{-}{}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}\text{-algebra}\}.$$

Let $\alpha \in \text{C}^*\text{-fact}(A; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$ and let ${}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}$ be a C^*-base. We put $\text{C}^*\text{-fact}(A_\alpha; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}) := \{\beta \in \text{C}^*\text{-fact}(A; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}) \mid \beta \perp \alpha\}$. If $\beta \in \text{C}^*\text{-fact}(A_\alpha; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger})$ (and A is nondegenerate), we call (H, A, α, β) a *(nondegenerate) concrete C^*-${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$-${}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}^\dagger}$-algebra.*

4

Let (H, A, α) be a concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra and (K, B, β) a concrete C^*-$_{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}}$-algebra. The *fiber product* of (H, A, α) and (K, B, β) is the C^*-algebra

$$A_\alpha \underset{\mathfrak{H}}{*}_\beta B := \big\{ T \in \mathcal{L}(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K) \,\big|\, T|\alpha\rangle_{[1]}, T^*|\alpha\rangle_{[1]} \subseteq [|\alpha\rangle_{[1]} B]$$

$$\text{and } T|\beta\rangle_{[2]}, T^*|\beta\rangle_{[2]} \subseteq [|\beta\rangle_{[2]} A] \big\}.$$

We do not know whether $A_\alpha \underset{\mathfrak{H}}{*}_\beta B$ is nondegenerate if A and B are nondegenerate. If $_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger}$ is a C^*-base, then $\gamma \triangleleft \beta \in C^*$-$\mathrm{fact}(A_\alpha \underset{\mathfrak{H}}{*}_\beta B; {}_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger})$ for each $\gamma \in C^*$-$\mathrm{fact}(A_\alpha; {}_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger})$ and $\alpha \triangleright \delta \in$ C^*-$\mathrm{fact}(A_\alpha \underset{\mathfrak{H}}{*}_\beta B; {}_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger})$ for each $\delta \in C^*$-$\mathrm{fact}(B_\beta; {}_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger})$.

We do not expect the fiber product to be associative. Given C^*-bases $_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$ and $_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger}$, a concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra (H, A, α), a concrete C^*-$_{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}}$-$_{\mathfrak{C}}\mathfrak{K}_{\mathfrak{C}^\dagger}$-algebra (K, B, β, γ) , and a concrete C^*-$_{\mathfrak{C}^\dagger}\mathfrak{K}_{\mathfrak{C}}$-algebra (L, C, δ), we put

$$A_\alpha \underset{\mathfrak{H}}{*}_\beta B_\gamma \underset{\mathfrak{K}}{*}_\delta C := \big((A_\alpha \underset{\mathfrak{H}}{*}_\beta B)_{\alpha \triangleright \gamma} \underset{\mathfrak{K}}{*}_\delta C\big) \cap \big(A_\alpha \underset{\mathfrak{H}}{*}_{\beta \triangleleft \delta} (B_\gamma \underset{\mathfrak{K}}{*}_\delta C)\big);$$

here, we canonically identify $(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K)_{\alpha \triangleright \gamma} \underset{\mathfrak{K}}{\otimes}_\delta L$ with $H_\alpha \underset{\mathfrak{H}}{\otimes}_{\beta \triangleleft \delta} (K_\gamma \underset{\mathfrak{K}}{\otimes}_\delta L)$.

Definition 2.1. *Let $_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$ be a C^*-base and (H, A, α), (K, B, β) concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebras. A* morphism *from (H, A, α) to (K, B, β) is a $*$-homomorphism $\pi \colon A \to B$ such that $\beta = [\mathcal{L}^\pi(H_\alpha, K_\beta)\alpha]$, where $\mathcal{L}^\pi(H_\alpha, K_\beta) := \big\{ V \in \mathcal{L}(H_\alpha, K_\beta) \,\big|\, \forall a \in A : \pi(a)V = Va \big\}$.*

We denote the set of all morphisms from (H, A, α) to (K, B, β) by $\mathrm{Mor}\big(A_\alpha, B_\beta\big)$. In [10, Definition 3.11], we imposed an additional condition on morphisms which is automatically satisfied:

Lemma 2.2. *Let $_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$ be a C^*-base and let π be a morphism of concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebras (H, A, α) and (K, B, β). Then $\pi(a\rho_\alpha(b^\dagger)) = \pi(a)\rho_\beta(b^\dagger)$ for all $a \in A$ and $b^\dagger \in \mathfrak{B}^\dagger$.*

Proof. Let $a \in A$ and $b^\dagger \in \mathfrak{B}^\dagger$. Then for all $V \in \mathcal{L}^\pi(H_\alpha, K_\beta)$, $\xi \in \alpha$, $\zeta \in \mathfrak{H}$,

$$\pi(a)\rho_\beta(b^\dagger)V\xi\zeta = \pi(a)V\xi b^\dagger \zeta = Va\rho_\alpha(b^\dagger)\xi\zeta = \pi(a\rho_\alpha(b^\dagger))V\xi\zeta.$$

Since $\big[\mathcal{L}^\pi(H_\alpha, K_\beta)\alpha\mathfrak{H}\big] = \big[\beta\mathfrak{H}\big] = K$, the claim follows. $\qquad\square$

Let π be a morphism of concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebras (H, A, α) and (K, B, β). Then $\big[\pi(A)K\big] = K$ [10, Remark 3.12].

Morphisms extend to multipliers as follows. Let (H, A, α) be a nondegenerate concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra. Then clearly also $(H, M(A), \alpha)$ is a concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra.

Definition 2.3. *Let $_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$ be a C^*-base and (H, A, α), (K, B, β) nondegenerate concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebras. We call a morphism π from (H, A, α) to $(K, M(B), \beta)$* nondegenerate *if $\big[\pi(A)B\big] = B$.*

Lemma 2.4. *Let $_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$ be a C^*-base, let (H, A, α), (K, B, β) be nondegenerate concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebras, and let π be a nondegenerate morphism from (H, A, α) to $(K, M(B), \beta)$. Then the unique strictly continuous extension $\tilde{\pi} \colon M(A) \to M(B)$ of π is a morphism from $(H, M(A), \alpha)$ to $(K, M(B), \beta)$.*

Proof. We only need to show that $\mathcal{L}^\pi(H_\alpha, K_\beta)$ is contained in $\mathcal{L}^{\tilde{\pi}}(H_\alpha, K_\beta)$. But if $V \in \mathcal{L}^\pi(H_\alpha, K_\beta)$ and $T \in M(A)$, then $VTa\zeta = \pi(Ta)V\zeta = \tilde{\pi}(T)\zeta\pi(a)V\zeta = \tilde{\pi}(T)Va\zeta$ for all $a \in A$ and $\zeta \in H$, and since $[AH] = H$, we can conclude $VT = \tilde{\pi}(T)V$. $\qquad\square$

Lemma 2.5. *Let (H, A, α) be a nondegenerate concrete C^*-$_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$-algebra and (K, B, β) a nondegenerate concrete C^*-$_{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}}$-algebra. Then $M(A)_\alpha \underset{\mathfrak{H}}{*}_\beta M(B) \subseteq M(A_\alpha \underset{\mathfrak{H}}{*}_\beta B)$.*

Proof. Let $T \in M(A)_\alpha \underset{\mathfrak{H}}{*}_\beta M(B)$. Then $T(A_\alpha \underset{\mathfrak{H}}{*}_\beta B) \subseteq A_\alpha \underset{\mathfrak{H}}{*}_\beta B$ because

$$T(A_\alpha \underset{\mathfrak{H}}{*}_\beta B)|\alpha\rangle_{[1]} \subseteq T\big[|\alpha\rangle_{[1]}B\big] \subseteq \big[|\alpha\rangle_{[1]}M(B)B\big] \subseteq \big[|\alpha\rangle_{[1]}B\big]$$

and similarly

$$(A_\alpha \underset{\mathfrak{H}}{*}_\beta B)T^*|\alpha\rangle_{[1]} \subseteq \big[|\alpha\rangle_{[1]}B\big], \qquad T(A_\alpha \underset{\mathfrak{H}}{*}_\beta B)|\beta\rangle_{[2]} \subseteq \big[|\beta\rangle_{[2]}B\big],$$
$$(A_\alpha \underset{\mathfrak{H}}{*}_\beta B)T^*|\beta\rangle_{[2]} \subseteq \big[|\beta\rangle_{[2]}B\big]. \qquad \qquad \square$$

Let ϕ be a morphism of nondegenerate concrete C^*-$_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$-algebras (H, A, α) and (L, γ, C), and let ψ be a morphism of nondegenerate concrete C^*-$_{\mathfrak{B}\dagger}\mathfrak{H}_\mathfrak{B}$-algebras (K, B, β) and (M, δ, D). Then there exists a unique $*$-homomorphism $\phi * \psi \colon A_\alpha \underset{\mathfrak{H}}{*}_\beta B \to C_\gamma \underset{\mathfrak{H}}{*}_\delta D$ such that $(\phi * \psi)(T) \cdot (X \underset{\mathfrak{H}}{\otimes} Y) = (X \underset{\mathfrak{H}}{\otimes} Y) \cdot T$ whenever $T \in A_\alpha \underset{\mathfrak{H}}{*}_\beta B$ and one of the following conditions holds: i) $X \in \mathcal{L}^\phi(H, L)$ and $Y \in \mathcal{L}^\psi(K_\beta, D_\delta)$ or ii) $X \in \mathcal{L}^\phi(H_\alpha, L_\gamma)$ and $Y \in \mathcal{L}^\psi(K, D)$. Moreover, let $_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger}$ be a C^*-base and assume that $A_\alpha \underset{\mathfrak{H}}{*}_\beta B \subseteq \mathcal{L}(H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta K)$ is nondegenerate. If $\alpha' \in C^*$-fact$(A_\alpha; _\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger})$, $\gamma' \in C^*$-fact$(C_\gamma; _\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger})$, $\phi \in \mathrm{Mor}(A_{\alpha'}, C_{\gamma'})$, then $\phi * \psi \in \mathrm{Mor}\big((A_\alpha \underset{\mathfrak{H}}{*}_\beta B)_{(\alpha' \lhd \beta)}, (C_\gamma \underset{\mathfrak{H}}{*}_\delta D)_{(\gamma' \lhd \delta)}\big)$. Similarly, if $\beta' \in C^*$-fact$(B_\beta; _\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger})$, $\delta' \in C^*$-fact$(D_\delta; _\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger})$, $\psi \in \mathrm{Mor}(B_{\beta'}, D_{\delta'})$, then $\phi * \psi \in \mathrm{Mor}\big((A_\alpha \underset{\mathfrak{H}}{*}_\beta B)_{(\alpha \rhd \beta')}, (C_\gamma \underset{\mathfrak{H}}{*}_\delta D)_{(\gamma \rhd \delta')}\big)$.

Lemma 2.6. *Let H be a Hilbert space, $\alpha \in C^*$-fact$(H; _{\mathfrak{B}\dagger}\mathfrak{H}_\mathfrak{B})$, and let ρ be a morphism of concrete C^*-$_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$-algebras (K, C, γ) and (L, D, δ). Then $\big[|\alpha\rangle_{[2]}C\langle\alpha|_{[2]}\big] \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{H}}{\otimes}_\alpha H)$ and $\big[|\alpha\rangle_{[2]}D\langle\alpha|_{[2]}\big] \subseteq \mathcal{L}(L_\delta \underset{\mathfrak{H}}{\otimes}_\alpha H)$ are C^*-algebras, and there exists a $*$-homomorphism $\mathrm{Ind}_\alpha(\rho) \colon \big[|\alpha\rangle_{[2]}C\langle\alpha|_{[2]}\big] \to \big[|\alpha\rangle_{[2]}D\langle\alpha|_{[2]}\big]$ such that for all $c \in C$ and $\xi, \xi' \in \alpha$,*

$$\mathrm{Ind}_\alpha(\rho)\big(|\xi\rangle_{[2]}c\langle\xi'|_{[2]}\big) = |\xi\rangle_{[2]}\rho(c)\langle\xi'|_{[2]}.$$

Proof. The space $\big[|\alpha\rangle_{[2]}C\langle\alpha|_{[2]}\big]$ is a C^*-algebra because

$$|\alpha\rangle_{[2]}C\langle\alpha|_{[2]}(|\alpha\rangle_{[2]}C\langle\alpha|_{[2]})^* \subseteq |\alpha\rangle_{[2]}C\rho_\gamma(\mathfrak{B})C\langle\alpha|_{[2]} \subseteq |\alpha\rangle_{[2]}C\langle\alpha|_{[2]}.$$

Likewise, $\big[|\alpha\rangle_{[2]}D\langle\alpha|_{[2]}\big]$ is a C^*-algebra. The existence of $\mathrm{Ind}_\alpha(\rho)$ follows as in the first part of the proof of [10, Proposition 3.13]. $\square$

A *concrete Hopf C^*-bimodule* is a tuple consisting of a C^*-base $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$, a nondegenerate concrete C^*-$_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$-$_{\mathfrak{B}\dagger}\mathfrak{H}_\mathfrak{B}$-algebra (H, A, α, β), and a $*$-homomorphism $\Delta \colon A \to A_\alpha \underset{\mathfrak{H}}{*}_\beta A$ subject to the following conditions:

i) $\Delta \in \mathrm{Mor}\big(A_\alpha, (A_\alpha \underset{\mathfrak{H}}{*}_\beta A)_{\alpha \rhd \alpha}\big) \cap \mathrm{Mor}\big(A_\beta, (A_\alpha \underset{\mathfrak{H}}{*}_\beta A)_{\beta \lhd \beta}\big)$, and

ii) the following diagram commutes:

$$\begin{array}{ccc}
A & \xrightarrow{\ \ \ \ \Delta\ \ \ \ } & A_\alpha \underset{\mathfrak{H}}{*}_\beta A \\
\Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\mathrm{id}*\Delta} \\
& & A_\alpha \underset{\mathfrak{H}}{*}_{\beta\lhd\beta}(A_\alpha \underset{\mathfrak{H}}{*}_\beta A) \\
& & \Big\uparrow \\
A_\alpha \underset{\mathfrak{H}}{*}_\beta A & \xrightarrow{\Delta*\mathrm{id}} (A_\alpha \underset{\mathfrak{H}}{*}_\beta A)_{\alpha\rhd\alpha}\underset{\mathfrak{H}}{*}_\beta A & \hookrightarrow \mathcal{L}(H_\alpha \underset{\mathfrak{H}}{*}_\beta H_\alpha \underset{\mathfrak{H}}{*}_\beta H).
\end{array}$$

If $(_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta)$ is a concrete Hopf C^*-bimodule, then the spaces $(\Delta*\mathrm{id})(\Delta(A))$ and $(\mathrm{id}*\Delta)(\Delta(A))$ are contained in $A_\alpha \underset{\mathfrak{H}}{*}_\beta A_\alpha \underset{\mathfrak{H}}{*}_\beta A$.

If $(_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta)$ is a concrete Hopf C^*-bimodule and $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$ and H are understood, we shall denote this concrete Hopf C^*-bimodule briefly by $(_\beta A_\alpha, \Delta)$.

6

C^*-pseudo-multiplicative unitaries and the associated legs The notion of a C^*-pseudo-multiplicative unitary extends the notion of a multiplicative unitary [1], of a continuous field of multiplicative unitaries [2], and of a pseudo-multiplicative unitary on C^*-modules [5, 11], and is closely related to pseudo-multiplicative unitaries on Hilbert spaces [12]; see [10, Section 4.1]. The precise definition is as follows.

Let H be a Hilbert space, $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ a C^*-base, $\alpha \in C^*\text{-fact}(H; _\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger})$, $\widehat{\beta} \in C^*\text{-fact}(H; _{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}})$, $\beta \in C^*\text{-fact}(H; _{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}})$ such that $\alpha, \beta, \widehat{\beta}$ are pairwise compatible. Let $V : H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H \to H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H$ be a unitary such that

$$V_*(\alpha \triangleleft \alpha) = \alpha \triangleright \alpha, \quad V_*(\widehat{\beta} \triangleright \beta) = \widehat{\beta} \triangleleft \beta, \quad V_*(\widehat{\beta} \triangleright \widehat{\beta}) = \alpha \triangleright \widehat{\beta}, \quad V_*(\beta \triangleleft \alpha) = \beta \triangleleft \beta. \qquad (4)$$

Then all operators in the following diagram are well-defined [10, Lemma 4.1],

$$(5)$$

where $\Sigma_{[23]}$ denotes the isomorphism

$$\big(H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H\big)_{\widehat{\beta}\triangleleft\beta}\underset{\mathfrak{H}}{\otimes}_\alpha H \cong (H_{\rho_\alpha}\oslash\beta)_{\rho_{\widehat{\beta}\triangleleft\beta}}\oslash\alpha \xrightarrow{\cong} (H_{\rho_{\widehat{\beta}}}\oslash\alpha)_{\rho_{\alpha\triangleleft\alpha}}\oslash\beta \cong (H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{H}}{\otimes}_\beta H,$$

$$(\zeta \oslash \xi) \oslash \eta \mapsto (\zeta \oslash \eta) \oslash \xi.$$

We put $V_{[12]} := V \underset{\mathfrak{H}}{\otimes} \text{id}$, $V_{[23]} := \text{id} \underset{\mathfrak{H}}{\otimes} V$, $V_{[13]} := (\text{id} \underset{\mathfrak{H}}{\otimes} \Sigma) V_{[12]}\Sigma_{[23]}$. We call V a C^*-*pseudo-multiplicative unitary* if Diagram (5) commutes, that is, if $V_{[12]}V_{[13]}V_{[23]} = V_{[23]}V_{[12]}$. In that case, also $V^{op} := \Sigma V^* \Sigma \colon H_\beta\underset{\mathfrak{H}}{\otimes}_\alpha H \to H_\alpha\underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}} H$ is a C^*-pseudo-multiplicative unitary, called the *opposite* of V.

Let $V \colon H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H \to H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H$ be a C^*-pseudo-multiplicative unitary. Then the spaces

$$\widehat{A} := \widehat{A}(V) := \big[\langle\beta|_{[2]} V|\alpha\rangle_{[2]}\big] \subseteq \mathcal{L}(H), \qquad A := A(V) := \big[\langle\alpha|_{[1]} V|\widehat{\beta}\rangle_{[1]}\big] \subseteq \mathcal{L}(H)$$

satisfy

$$\big[\widehat{A}\widehat{A}\big] = \big[\widehat{A}\rho_{\widehat{\beta}}(\mathfrak{B})\big] = \big[\rho_{\widehat{\beta}}(\mathfrak{B})\widehat{A}\big] = \big[\widehat{A}\rho_\alpha(\mathfrak{B}^\dagger)\big] = \big[\rho_\alpha(\mathfrak{B}^\dagger)\widehat{A}\big] = \widehat{A} \subseteq \mathcal{L}(H_\beta),$$
$$\big[AA\big] = \big[A\rho_\beta(\mathfrak{B})\big] = \big[\rho_\beta(\mathfrak{B})A\big] = \big[A\rho_\alpha(\mathfrak{B}^\dagger)\big] = \big[\rho_\alpha(\mathfrak{B}^\dagger)A\big] = A \subseteq \mathcal{L}(H_{\widehat{\beta}}).$$

Define maps

$$\widehat{\Delta}_V \colon \widehat{A} \to \mathcal{L}\big(H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H\big), \; y \mapsto V^*(1 \underset{\mathfrak{H}}{\otimes} y)V, \quad \Delta_V \colon A \to \mathcal{L}\big(H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H\big), \; z \mapsto V(z \underset{\mathfrak{H}}{\otimes} 1)V^*.$$

We call V *well-behaved* if the tuples $\big(_{\mathfrak{B}^\dagger}\mathfrak{H}_\mathfrak{B}, H, \widehat{A}, \widehat{\beta}, \alpha, \widehat{\Delta}_V\big)$ and $\big(_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}, H, A, \alpha, \beta, \Delta_V\big)$ are concrete Hopf C^*-bimodules. We call V *regular* if the subspace $\big[\langle\alpha|_{[1]} V|\alpha\rangle_{[2]}\big] \subseteq \mathcal{L}(H)$ is equal to $[\alpha\alpha^*]$. If V is regular, then it is well-behaved [10, Theorem 4.14].

7

3   Weak C^*-pseudo-Kac systems

Reduced crossed products for coactions of Hopf C^*-algebras can conveniently be constructed in the framework of Kac systems [1] or, more generally, of weak Kac systems [13]. To adapt the construction to coactions of concrete Hopf C^*-bimodules, we generalize the notion of a weak Kac system as follows.

Let H be a Hilbert space. Recall that for each C^*-base ${}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger}$ and each C^*-factorization $\gamma \in \mathrm{C}^*\text{-fact}(H; {}_\mathfrak{C}\mathfrak{K}_{\mathfrak{C}\dagger})$, $\delta \in \mathrm{C}^*\text{-fact}(H; {}_{\mathfrak{C}\dagger}\mathfrak{K}_\mathfrak{C})$, there exists a flip map

$$\Sigma \colon H_\gamma \underset{\mathfrak{K}}{\otimes}_\delta H = \gamma \oslash \mathfrak{K} \oslash \delta \to \delta \oslash \mathfrak{K} \oslash \gamma = H_\delta \underset{\mathfrak{K}}{\otimes}_\gamma H, \quad \eta \oslash \zeta \oslash \xi \mapsto \xi \oslash \zeta \oslash \eta.$$

Let ${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$ be a C^*-base, $\alpha, \widehat{\alpha} \in \mathrm{C}^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger})$, $\beta, \widehat{\beta} \in \mathrm{C}^*\text{-fact}(H; {}_{\mathfrak{B}\dagger}\mathfrak{H}_\mathfrak{B})$, and let $U \colon H \to H$ be a symmetry, that is, a self-adjoint unitary. Assume that $U\alpha = \widehat{\alpha}$ and $U\beta = \widehat{\beta}$; then also $U\widehat{\alpha} = \alpha$ and $U\widehat{\beta} = \beta$. For each $T \in \mathcal{L}(H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H, H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H)$, put

$$\check{T} := \Sigma(1 \underset{\mathfrak{H}}{\otimes} U)T(1 \underset{\mathfrak{H}}{\otimes} U)\Sigma \colon H_{\widehat{\alpha}} \underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}} H \xrightarrow{U_{[2]}\Sigma} H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H \xrightarrow{T} H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H \xrightarrow{\Sigma U_{[2]}} H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H,$$

$$\widehat{T} := \Sigma(U \underset{\mathfrak{H}}{\otimes} 1)T(U \underset{\mathfrak{H}}{\otimes} 1)\Sigma \colon H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H \xrightarrow{U_{[1]}\Sigma} H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H \xrightarrow{T} H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H \xrightarrow{\Sigma U_{[1]}} H_\beta \underset{\mathfrak{H}}{\otimes}_{\widehat{\alpha}} H.$$

Switching between the C^*-bases ${}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger}$ and ${}_{\mathfrak{B}\dagger}\mathfrak{H}_\mathfrak{B}$ and relabeling the C^*-factorizations $\alpha, \widehat{\alpha}, \beta, \widehat{\beta}$ suitably, we can iterate the maps $T \mapsto \check{T}$ and $T \mapsto \widehat{T}$. The two relations $\Sigma(1 \underset{\mathfrak{H}}{\otimes} U)\Sigma(1 \underset{\mathfrak{H}}{\otimes} U) = U \underset{\mathfrak{H}}{\otimes} U = \Sigma(U \underset{\mathfrak{H}}{\otimes} 1)\Sigma(U \underset{\mathfrak{H}}{\otimes} 1)$ and $\Sigma(1 \underset{\mathfrak{H}}{\otimes} U)\Sigma(U \underset{\mathfrak{H}}{\otimes} 1) = \mathrm{id}$ imply

$$\check{\check{T}} = \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} U)}(T) = \widehat{\widehat{T}}, \qquad \check{\widehat{\check{T}}} = \widehat{T}, \qquad \check{\widehat{T}} = \widehat{\check{T}} \qquad \text{for all } T \in \mathcal{L}(H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H, H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H). \tag{6}$$

Definition 3.1. *A* balanced C^*-pseudo-multiplicative unitary $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *consists of*

- *C^*-factorizations $\alpha, \widehat{\alpha} \in \mathrm{C}^*\text{-fact}(H; {}_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}\dagger})$ and $\beta, \widehat{\beta} \in \mathrm{C}^*\text{-fact}(H; {}_{\mathfrak{B}\dagger}\mathfrak{H}_\mathfrak{B})$,*
- *a symmetry $U \colon H \to H$, and*
- *a C^*-pseudo-multiplicative unitary $V \colon H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H \to H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H$*

satisfying the following conditions:

 i) $\alpha, \widehat{\alpha}, \beta, \widehat{\beta}$ *are pairwise compatible,*

 ii) $U\alpha = \widehat{\alpha}$ *and* $U\beta = \widehat{\beta}$,

 iii) $\check{V}$ *and* $\widehat{V}$ *are C^*-pseudo-multiplicative unitaries.*

Remarks 3.2.   i) Since $\widehat{V} = (U \underset{\mathfrak{H}}{\otimes} U)\check{V}(U \underset{\mathfrak{H}}{\otimes} U)$, the unitary $\widehat{V}$ is C^*-pseudo-multiplicative if and only if $\check{V}$ is C^*-pseudo-multiplicative.

ii) If $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ is a balanced C^*-pseudo-multiplicative unitary, then also the tuples $(\widehat{\beta}, \beta, \alpha, \widehat{\alpha}, U, \check{V})$, $(\widehat{\alpha}, \alpha, \widehat{\beta}, \beta, U, \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} U)}(V))$, and $(\beta, \widehat{\beta}, \widehat{\alpha}, \alpha, U, \widehat{V})$ are balanced C^*-pseudo-multiplicative unitaries. This follows easily from (6). Moreover, in that case $(\alpha, \widehat{\alpha}, \widehat{\beta}, \beta, U, V^{op})$ is a balanced C^*-pseudo-multiplicative unitary as well, as can be seen from the relation

$$\widetilde{(V^{op})} = \Sigma U_{[2]}(\Sigma V^* \Sigma)U_{[2]}\Sigma = \Sigma(\Sigma U_{[1]} V U_{[1]}\Sigma)^* \Sigma = (\widehat{V})^{op}. \tag{7}$$

iii) If $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ is a balanced C^*-pseudo-multiplicative unitary, the relations (4) for the unitaries $\check{V} \colon H_{\widehat{\alpha}} \underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}} H \to H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H$ and $\widehat{V} \colon H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H \to H_\beta \underset{\mathfrak{H}}{\otimes}_{\widehat{\alpha}} H$ read as follows:

$$\check{V}_*(\widehat{\beta} \triangleleft \widehat{\beta}) = \widehat{\beta} \triangleright \widehat{\beta}, \quad \check{V}_*(\widehat{\alpha} \triangleright \alpha) = \widehat{\alpha} \triangleleft \alpha, \quad \check{V}_*(\widehat{\alpha} \triangleright \widehat{\alpha}) = \widehat{\beta} \triangleright \widehat{\alpha}, \quad \check{V}_*(\alpha \triangleleft \widehat{\beta}) = \alpha \triangleleft \alpha,$$

$$\widehat{V}_*(\beta \triangleleft \beta) = \beta \triangleright \beta, \quad \widehat{V}_*(\alpha \triangleright \widehat{\alpha}) = \alpha \triangleleft \widehat{\alpha}, \quad \widehat{V}_*(\alpha \triangleright \alpha) = \beta \triangleright \alpha, \quad \widehat{V}_*(\widehat{\alpha} \triangleleft \beta) = \widehat{\alpha} \triangleleft \widehat{\alpha}.$$

These relations furthermore imply

$$V_*(\hat{\beta} \triangleright \hat{\alpha}) = \alpha \triangleright \hat{\alpha}, \qquad \check{V}_*(\hat{\alpha} \triangleright \beta) = \hat{\beta} \triangleright \beta, \qquad \hat{V}_*(\alpha \triangleright \hat{\beta}) = \beta \triangleright \hat{\beta},$$
$$V_*(\hat{\alpha} \triangleleft \alpha) = \hat{\alpha} \triangleleft \beta, \qquad \check{V}_*(\beta \triangleleft \hat{\beta}) = \beta \triangleleft \alpha, \qquad \hat{V}_*(\hat{\beta} \triangleleft \beta) = \hat{\beta} \triangleleft \hat{\alpha}.$$

iv) If $(\alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)$ is a balanced C^*-pseudo-multiplicative unitary, then $\hat{A}(V), A(V) \subseteq \mathcal{L}(H_{\hat{\alpha}})$ because

$$\left[\hat{A}(V)\hat{\alpha}\right] = \left[\langle\beta|_{[2]}V|\alpha\rangle_{[2]}\hat{\alpha}\right] = \left[\langle\beta|_{[2]}|\beta\rangle_{[2]}\hat{\alpha}\right] = \left[\rho_\alpha(\mathfrak{B}^\dagger)\hat{\alpha}\right] = \hat{\alpha} \qquad (8)$$

and similarly $\left[A(V)\hat{\alpha}\right] = \left[\langle\alpha|_{[1]}V|\hat{\beta}\rangle_{[1]}\hat{\alpha}\right] = \left[\langle\alpha|_{[1]}|\alpha\rangle_{[1]}\hat{\alpha}\right] = \left[\rho_\beta(\mathfrak{B})\hat{\alpha}\right] = \hat{\alpha}$.

The auxiliary unitaries $\check{V}$ and $\hat{V}$ defined above allow us to treat the right and the left leg of V, respectively, as the left or the right leg of some C^*-pseudo-multiplicative unitary:

Proposition 3.3. *Let $(\alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)$ be a balanced C^*-pseudo-multiplicative unitary. Then*

$$\hat{A}(\check{V}) = \mathrm{Ad}_U(A(V)), \quad \hat{\Delta}_{\check{V}} = \mathrm{Ad}_{(U\underset{\mathfrak{H}}{\otimes}U)} \circ \Delta_V \circ \mathrm{Ad}_U, \qquad A(\check{V}) = \hat{A}(V), \quad \Delta_{\check{V}} = \hat{\Delta}_V,$$
$$A(\hat{V}) = \mathrm{Ad}_U(\hat{A}(V)), \quad \Delta_{\hat{V}} = \mathrm{Ad}_{(U\underset{\mathfrak{H}}{\otimes}U)} \circ \hat{\Delta}_V \circ \mathrm{Ad}_U, \qquad \hat{A}(\hat{V}) = A(V), \quad \hat{\Delta}_{\hat{V}} = \Delta_V.$$

In particular, $\check{V}$ and $\hat{V}$ are well-behaved if V is well-behaved.

For the proof, we need the following lemma:

Lemma 3.4. *The following diagrams commute:*

$$(H_{\hat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\hat{\beta}}H)_{\hat{\beta}\triangleleft\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H \xrightarrow{\ V_{[13]}\ } (H_{\hat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\hat{\beta}}H)_{\alpha\triangleleft\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\beta H \xrightarrow{\ \check{V}_{[12]}\ } (H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{H}}{\otimes}_\beta H \qquad (9)$$

with vertical arrows $\check{V}_{[12]}\downarrow$ on the left and $\uparrow V_{[13]}$ on the right, and bottom

$$H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H \xrightarrow{\hspace{4cm} V_{[23]}\hspace{4cm}} H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha\triangleright\alpha}(H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H)$$

and

$$H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha\triangleright\alpha}(H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H) \xrightarrow{\ \hat{V}_{[23]}\ } H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\beta\triangleright\alpha}(H_\beta\underset{\mathfrak{H}}{\otimes}_{\hat{\alpha}}H) \xrightarrow{\ V_{[13]}\ } H_\alpha\underset{\mathfrak{H}}{\otimes}_{\beta\triangleright\beta}(H_\beta\underset{\mathfrak{H}}{\otimes}_{\hat{\alpha}}H). \qquad (10)$$

with vertical arrows $V_{[13]}\downarrow$ on the left and $\uparrow\hat{V}_{[23]}$ on the right, and bottom

$$(H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H)_{\alpha\triangleleft\alpha}\underset{\mathfrak{H}}{\otimes}_\beta H \xrightarrow{\hspace{4cm} V_{[12]}\hspace{4cm}} H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H$$

Proof. Let us prove that diagram (9) commutes. Put $W := \Sigma V \Sigma$. We insert the relation $\check{V} = U_{[1]}WU_{[1]}$ into the pentagon equation $\check{V}_{[12]}\check{V}_{[13]}\check{V}_{[23]} = \check{V}_{[23]}\check{V}_{[12]}$ and obtain

$$U_{[1]}W_{[12]}U_1 \cdot U_1 W_{[13]}U_1 \cdot \check{V}_{[23]} = \check{V}_{[23]} \cdot U_{[1]}W_{[12]}U_{[1]}.$$

Since $U_{[1]}$ commutes with $\check{V}_{[23]}$, we can cancel $U_{[1]}$ everywhere in the equation above and find $W_{[12]}W_{[13]}\check{V}_{[23]} = \check{V}_{[23]}W_{[12]}$. Now, we conjugate both sides of this equation by the automorphism $\Sigma_{[23]}\Sigma_{[12]}$, which amounts to renumbering the legs of the operators according to the permutation $(1, 2, 3) \mapsto (2, 3, 1)$, and find $V_{[13]}V_{[23]}\check{V}_{[12]} = \check{V}_{[12]}V_{[13]}$. If we retrace the derivation of this equation in diagrammatic form, we obtain diagram (9).

A similar argument shows that diagram (10) commutes. $\qquad\square$

Proof of Proposition 3.3. Inserting the relations

$$\check{V} = \Sigma U_{[2]}VU_{[2]}\Sigma\colon H_{\hat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\hat{\beta}}H \to H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H, \quad \hat{V} = \Sigma U_{[1]}VU_{[1]}\Sigma\colon H_\alpha\underset{\mathfrak{H}}{\otimes}_\beta H \to H_\beta\underset{\mathfrak{H}}{\otimes}_{\hat{\alpha}}H$$

9

into the definition of $A(\check{V})$ and $\widehat{A}(\widehat{V})$, respectively, we find

$$A(\check{V}) = \big[\langle\widehat{\beta}|_{[1]}\Sigma U_{[2]}VU_{[2]}\Sigma|\widehat{\alpha}\rangle_{[1]}\big] = \big[\langle U\widehat{\beta}|_{[2]}V|U\widehat{\alpha}\rangle_{[2]}\big] = \big[\langle\beta|_{[2]}V|\alpha\rangle_{[2]}\big] = \widehat{A}(V),$$

$$\widehat{A}(\widehat{V}) = \big[\langle\widehat{\alpha}|_{[2]}\Sigma U_{[1]}VU_{[1]}\Sigma|\beta\rangle_{[2]}\big] = \big[\langle U\widehat{\alpha}|_{[1]}V|U\beta\rangle_{[1]}\big] = \big[\langle\alpha|_{[1]}V|\widehat{\beta}\rangle_{[1]}\big] = A(V).$$

To prove $\Delta_{\check{V}} = \widehat{\Delta}_V$, consider an element $\widehat{a} = \langle\xi'|_{[2]}V|\xi\rangle_{[2]} \in \widehat{A}(V)$, where $\xi' \in \beta$, $\xi \in \alpha$. By definition, $\Delta_{\check{V}}(\widehat{a}) = \check{V}(\widehat{a}\underset{\mathfrak{H}}{\otimes}1)\check{V}^*$. The commutative diagram

$$
\begin{array}{ccccccc}
H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{\;\;\check{V}^*\;\;} & H_{\widehat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}}H & \xrightarrow{\;\;\widehat{a}\underset{\mathfrak{H}}{\otimes}1\;\;} & H_{\widehat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}}H & \xrightarrow{\;\;\check{V}\;\;} & H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H, \\
\big\downarrow{\scriptstyle|\xi\rangle_{[3]}} & & \big\downarrow{\scriptstyle|\xi\rangle_{[3]}} & & \big\uparrow{\scriptstyle\langle\xi'|_{[3]}} & & \big\uparrow{\scriptstyle\langle\xi'|_{[3]}} \\
H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{\check{V}_{[12]}^*} & (H_{\widehat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}}H)_{\widehat{\beta}\vartriangleleft\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{V_{[13]}} & (H_{\widehat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\widehat{\beta}}H)_{\alpha\vartriangleleft\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\beta}H & \xrightarrow{\check{V}_{[12]}} & (H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H)_{\alpha\vartriangleleft\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H
\end{array}
$$

diagram (9) and the pentagon diagram (5) imply

$$\check{V}(\widehat{a}\underset{\mathfrak{H}}{\otimes}1)\check{V}^* = \langle\xi'|_{[3]}\check{V}_{[12]}V_{[13]}\check{V}_{[12]}^*|\xi\rangle_{[3]}$$

$$= \langle\xi'|_{[3]}V_{[13]}V_{[23]}|\xi\rangle_{[3]} = \langle\xi'|_{[3]}V_{[12]}^*V_{[23]}V_{[12]}|\xi\rangle_{[3]}.$$

The following diagram shows that the expression above is equal to $V^*(1\underset{\mathfrak{H}}{\otimes}\widehat{a})V = \widehat{\Delta}_V(\widehat{a})$:

$$
\begin{array}{ccccccc}
H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{\;\;V\;\;} & H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H & \xrightarrow{\;\;1\underset{\mathfrak{H}}{\otimes}\widehat{a}\;\;} & H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H & \xrightarrow{\;\;V^*\;\;} & H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H. \\
\big\downarrow{\scriptstyle|\xi\rangle_{[3]}} & & \big\downarrow{\scriptstyle|\xi\rangle_{[3]}} & & \big\uparrow{\scriptstyle\langle\xi'|_{[3]}} & & \big\uparrow{\scriptstyle\langle\xi'|_{[3]}} \\
H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{V_{[12]}} & H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{V_{[23]}} & H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H & \xrightarrow{V_{[12]}^*} & (H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H)_{\alpha\vartriangleleft\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H
\end{array}
$$

Since elements of the form like $\widehat{a}$ are dense in $\widehat{A}(V)$, we can conclude $\Delta_{\check{V}} = \widehat{\Delta}_{\check{V}}$.

A similar argument shows that $\widehat{\Delta}_{\widehat{V}} = \Delta_V$; here, we have to use diagram (10).

The remaining equations in Proposition 3.3 follow from the relation $\widehat{V} = \mathrm{Ad}_{(U\underset{\mathfrak{H}}{\otimes}U)}(\check{V})$ and those equations that we have proved already. $\qquad\square$

The definition of a weak C^*-pseudo-Kac system involves the following conditions:

Lemma 3.5. *Let $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ be a balanced C^*-pseudo-multiplicative unitary.*

i) The following conditions are equivalent:

(a) The following diagram commutes:
$$
\begin{array}{ccc}
H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{\;\widehat{V}_{[12]}\;} & H_{\beta}\underset{\mathfrak{H}}{\otimes}_{\widehat{\alpha}}H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H \\
\big\downarrow{\scriptstyle V_{[23]}} & & \big\downarrow{\scriptstyle V_{[23]}} \\
H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H & \xrightarrow{\;\widehat{V}_{[12]}\;} & H_{\beta}\underset{\mathfrak{H}}{\otimes}_{\widehat{\alpha}}H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H.
\end{array}
$$

(b) $(1\underset{\mathfrak{H}}{\otimes}\widehat{a})\widehat{V} = \widehat{V}(1\underset{\mathfrak{H}}{\otimes}\widehat{a})$ in $\mathcal{L}(H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H, H_{\beta}\underset{\mathfrak{H}}{\otimes}_{\widehat{\alpha}}H)$ for each $\widehat{a}\in\widehat{A}(V)$.

(c) $(\mathrm{Ad}_U(\widehat{a})\underset{\mathfrak{H}}{\otimes}1)V = V(\mathrm{Ad}_U(\widehat{a})\underset{\mathfrak{H}}{\otimes}1)$ in $\mathcal{L}(H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H, H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H)$ for each $\widehat{a}\in\widehat{A}(V)$.

(d) $\widehat{A}(V)$ and $\mathrm{Ad}_U(\widehat{A}(V))$ commute.

ii) The following conditions are equivalent:

10

(a) *The following diagram commutes:*

$$
\begin{array}{ccc}
H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H_{\hat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\hat{\beta}}H & \xrightarrow{V_{[12]}} & H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\hat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\hat{\beta}}H \\
\downarrow{\scriptstyle\check{V}_{[23]}} & & \downarrow{\scriptstyle\check{V}_{[23]}} \\
H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H & \xrightarrow{V_{[12]}} & H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H.
\end{array}
$$

(b) $(a\underset{\mathfrak{H}}{\otimes}1)\check{V} = \check{V}(a\underset{\mathfrak{H}}{\otimes}1)$ *in* $\mathcal{L}(H_{\hat{\alpha}}\underset{\mathfrak{H}}{\otimes}_{\hat{\beta}}H, H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H)$ *for each* $a \in A(V)$.

(c) $(1\underset{\mathfrak{H}}{\otimes}\mathrm{Ad}_U(a))V = V(1\underset{\mathfrak{H}}{\otimes}\mathrm{Ad}_U(a))$ *in* $\mathcal{L}(H_{\hat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H, H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H)$ *for each* $a \in A(V)$.

(d) $A(V)$ *and* $\mathrm{Ad}_U(A(V))$ *commute.*

Proof. In i), conditions (a) and (b) are equivalent because

$$
\begin{aligned}
(a) &\Leftrightarrow \forall \xi \in \alpha, \xi' \in \beta : \langle\xi'|_{[3]}V_{[23]}\widehat{V}_{[12]}|\xi\rangle_{[3]} = \langle\xi'|_{[3]}\widehat{V}_{[12]}V_{[23]}|\xi\rangle_{[3]} \\
&\Leftrightarrow \forall \xi \in \alpha, \xi' \in \beta : (1\underset{\mathfrak{H}}{\otimes}\widehat{a})\widehat{V} = \widehat{V}(1\underset{\mathfrak{H}}{\otimes}\widehat{a}), \text{ where } \widehat{a} = \langle\xi'|_{[2]}V|\xi\rangle_{[2]} \Leftrightarrow (b).
\end{aligned}
$$

The remaining equivalences follow similarly. $\qquad\square$

Definition 3.6. *We call a balanced C^*-pseudo-multiplicative unitary $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ a weak C^*-pseudo-Kac system if V is well-behaved and if the equivalent conditions in Lemma 3.5 hold.*

The notion of a weak C^*-pseudo-Kac system is symmetric in the following sense:

Proposition 3.7. *Let $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ be a weak C^*-pseudo-Kac system. Then also the following tuples are weak C^*-pseudo-Kac systems:*

$$
(\widehat{\beta}, \beta, \alpha, \widehat{\alpha}, U, \check{V}), \quad (\beta, \widehat{\beta}, \widehat{\alpha}, \alpha, U, \widehat{V}), \quad (\alpha, \widehat{\alpha}, \widehat{\beta}, \beta, U, V^{op}). \tag{11}
$$

Proof. The C^*-pseudo-multiplicative unitaries $\check{V}$, $\widehat{V}$, V^{op} are well-behaved by Proposition 3.3 and [10, Lemma 4.4], respectively, and the tuples (11) are balanced C^*-pseudo-multiplicative unitaries by Remark 3.2 ii). Using Proposition 3.3 and [10, Lemma 4.4], one easily checks that these tuples satisfy the conditions i)(d) and ii)(d) of Lemma 3.5. $\qquad\square$

Definition 3.8. *Given a weak C^*-pseudo-Kac system $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$, we call the tuples (11) its* predual, *its* dual, *and its* opposite, *respectively.*

4 Coactions and reduced crossed products

Coactions of concrete Hopf C^*-bimodules

Definition 4.1. *Let $({}_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta)$ be a concrete Hopf C^*-bimodule and (K, C, γ) a nondegenerate concrete C^*-${}_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}\dagger}$-algebra. Then a coaction of $({}_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta)$ on (K, C, γ) is a morphism δ_C from (K, C, γ) to $(K_{\gamma}\underset{\mathfrak{H}}{\otimes}_{\beta}H, C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A), \gamma \triangleright \alpha)$ that makes the following diagram commute:*

$$
\begin{array}{ccc}
C & \xrightarrow{\quad\delta_C\quad} & C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A) \\
\downarrow{\scriptstyle\delta_C} & & \downarrow{\scriptstyle\delta_C *\mathrm{id}_{M(A)}} \\
& & (C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A))_{\gamma\triangleright\alpha}\underset{\mathfrak{H}}{*}_{\beta}M(A) \\
& & \curvearrowright \\
C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A) \xrightarrow{\mathrm{id}_C *\Delta} C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta\triangleleft\beta}M(A_{\alpha}\underset{\mathfrak{H}}{*}_{\beta}A) \hookrightarrow & \mathcal{L}(K_{\gamma}\underset{\mathfrak{H}}{\otimes}_{\beta}H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H).
\end{array}
$$

11

We also refer to the tuple (K, C, γ, δ_C) as a coaction. We call such a coaction *fine* if δ_C is injective and $\left[\delta_C(C)|\beta\rangle_{[2]}\right] = \left[|\beta\rangle_{[2]}C\right]$ as subsets of $\mathcal{L}(K, K_{\gamma}\underset{\mathfrak{H}}{\otimes}_{\beta}H)$.

A covariant morphism *between coactions* (K, C, γ, δ_C) and (L, D, δ, δ_D) *is a morphism* ρ *from* (K, C, γ) *to* $(L, M(D), \delta)$ *that makes the following diagram commute:*

$$
\begin{array}{ccc}
C & \xrightarrow{\quad\rho\quad} & M(D) \\
\downarrow{\scriptstyle \delta_C} & & \downarrow{\scriptstyle \delta_D} \\
C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A) & \xrightarrow{\rho * \mathrm{id}_{M(A)}} & M(D_{\delta}\underset{\mathfrak{H}}{*}_{\beta}A).
\end{array}
$$

Remarks 4.2.   i) Note that by [10, Remark 3.12], the C^*-algebra $\delta_C(C)$ and hence also the C^*-algebra $C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}A$ is nondegenerate.

   ii) Evidently, the class of all coactions of a fixed concrete Hopf-C^*-bimodule forms a category with respect to covariant morphisms.

  iii) For every concrete Hopf C^*-bimodule $(_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta)$, the triple (H, A, α, Δ) is a coaction.

We shall study coactions of concrete Hopf C^*-bimodules in a separate article.

Reduced crossed products for coactions of $\left(_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta_V\right)$ Till the end of this section, we fix a weak C^*-pseudo-Kac system $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$. To shorten the notation, we put

$$\widehat{A} := \widehat{A}(V), \qquad \widehat{\Delta} := \widehat{\Delta}_V, \qquad A := A(V), \qquad \Delta := \Delta_V,$$

and write $(_{\alpha}\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$ and $(_{\beta}A_{\alpha}, \Delta)$ for the concrete Hopf C^*-bimodules $\left(_{\mathfrak{B}\dagger}\mathfrak{H}_{\mathfrak{B}}, H, \widehat{A}, \widehat{\beta}, \alpha, \widehat{\Delta}\right)$ and $\left(_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}\dagger}, H, A, \alpha, \beta, \Delta\right)$, respectively.

First, we define reduced crossed products for coactions of $(_{\beta}A_{\alpha}, \Delta)$:

Definition 4.3. *Let* (K, C, γ, δ_C) *be a coaction of* $(_{\beta}A_{\alpha}, \Delta)$. *The associated* reduced crossed product *is the* C^*-subalgebra $C \rtimes_{\delta_C, r} \widehat{A} \subseteq \mathcal{L}(K_{\gamma}\underset{\mathfrak{H}}{\otimes}_{\beta}H)$ *generated by*

$$\delta(C)\left(1\underset{\mathfrak{H}}{\otimes}\widehat{A}\right) \subseteq \mathcal{L}(K_{\gamma}\underset{\mathfrak{H}}{\otimes}_{\beta}H). \tag{12}$$

If δ_C *is understood, we shortly write* $C \rtimes_r \widehat{A}$ *for* $C \rtimes_{\delta_C, r} \widehat{A}$.

Remark 4.4. In the situation above, $C \rtimes_r \widehat{A} \subseteq C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}\mathcal{L}(H)$, as can be seen from the inclusions $\delta_C(C) \subseteq C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A)$ and

$$\left(C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A)\right)\left(1\underset{\mathfrak{H}}{\otimes}\widehat{A}\right)|\gamma\rangle_{[1]} = \left(C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A)\right)|\gamma\rangle_{[1]}\widehat{A} \subseteq \left[|\gamma\rangle_{[1]}M(A)\widehat{A}\right],$$

$$\left(C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A)\right)\left(1\underset{\mathfrak{H}}{\otimes}\widehat{A}\right)|\beta\rangle_{[2]} \subseteq \left[\left(C_{\gamma}\underset{\mathfrak{H}}{*}_{\beta}M(A)\right)|\beta\rangle_{[2]}\right] \subseteq \left[|\beta\rangle_{[2]}C\right];$$

here, we used the inclusion $\widehat{A}\beta \subseteq \beta$ [10, Lemma 4.5].

Frequently, it is useful to know that the set (12) is linearly dense in $C \rtimes_r \widehat{A}$:

Proposition 4.5. *Let* (K, C, γ, δ_C) *be a coaction of* $(_{\beta}A_{\alpha}, \Delta)$. *Then*

$$C \rtimes_r \widehat{A} = \left[\delta_C(C)\left(1\underset{\mathfrak{H}}{\otimes}\widehat{A}\right)\right].$$

Proof. We only need to prove $\big[(1 \underset{\mathfrak{H}}{\otimes} \widehat{A})\delta_C(C)\big] \subseteq \big[\delta_C(C)(1 \underset{\mathfrak{H}}{\otimes} \widehat{A})\big]$. By definition of $\widehat{A}$,

$$\big[(1 \underset{\mathfrak{H}}{\otimes} \widehat{A})\delta_C(C)\big] = \big[\langle\beta|_{[3]}(1 \underset{\mathfrak{H}}{\otimes} V)|\alpha\rangle_{[3]}\delta_C(C)\big] = \big[\langle\beta|_{[3]}(1 \underset{\mathfrak{H}}{\otimes} V)(\delta_C(C) \underset{\mathfrak{H}}{\otimes} 1)|\alpha\rangle_{[3]}\big].$$

Note that $\delta_C(C) \underset{\mathfrak{H}}{\otimes} 1 \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{H}}{\otimes}_\beta H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H)$ is well-defined because $\delta_C(C) \subseteq C_{\gamma} \underset{\mathfrak{H}}{*}_\beta M(A)$ commutes with $1 \underset{\mathfrak{H}}{\otimes} \rho_{\widehat{\beta}}(\mathfrak{B})$ by [10, Lemma 3.8]. Put $\delta_C^{(2)} := (\mathrm{id} * \Delta) \circ \delta$. By definition of Δ,

$$\big[\langle\beta|_{[3]}(1 \underset{\mathfrak{H}}{\otimes} V)(\delta_C(C) \underset{\mathfrak{H}}{\otimes} 1)|\alpha\rangle_{[3]}\big] = \big[\langle\beta|_{[3]}(\delta_C^{(2)}(C))(1 \underset{\mathfrak{H}}{\otimes} V)|\alpha\rangle_{[3]}\big].$$

Since $\delta_C^{(2)}(C) = (\mathrm{id} * \Delta)(\delta_C(C)) = (\delta_C * \mathrm{id})(\delta_C(C)) \subseteq \delta_C(C)_{\gamma \triangleright \alpha} \underset{\mathfrak{H}}{*}_\beta M(A)$,

$$\big[\langle\beta|_{[3]}(\delta_C^{(2)}(C))(1 \underset{\mathfrak{H}}{\otimes} V)|\alpha\rangle_{[3]}\big] \subseteq \big[\delta_C(C)\langle\beta|_{[3]}(1 \underset{\mathfrak{H}}{\otimes} V)|\alpha\rangle_{[3]}\big] = \big[\delta_C(C)(1 \underset{\mathfrak{H}}{\otimes} \widehat{A})\big]. \qquad \square$$

Corollary 4.6. *Let (K, C, γ, δ_C) be a coaction of $(_\beta A_\alpha, \Delta)$. Then $\delta_C(C)$ and $1 \underset{\mathfrak{H}}{\otimes} \widehat{A}$ are nondegenerate C^*-subalgebras of $M(C \rtimes_r \widehat{A})$, and the maps $c \mapsto \delta_C(c)$ and $\widehat{a} \mapsto 1 \underset{\mathfrak{H}}{\otimes} \widehat{a}$ extend to $*$-homomorphisms $M(C) \to M(C \rtimes_r \widehat{A})$ and $M(\widehat{A}) \to M(C \rtimes_r \widehat{A})$, respectively.* $\qquad \square$

The reduced crossed product $C \rtimes_r \widehat{A}$ carries a dual coaction $\widehat{\delta}_C$ of $(_\alpha\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$:

Theorem 4.7. *Let (K, C, γ, δ_C) be a coaction of $(_\beta A_\alpha, \Delta)$. Then there exists a coaction*

$$\big(K_\gamma \underset{\mathfrak{H}}{\otimes}_\beta H, \; C \rtimes_r \widehat{A}, \; \gamma \triangleright \widehat{\beta}, \; \widehat{\delta}_C\big) \tag{13}$$

of the concrete Hopf C^*-bimodule $(_\alpha\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$ such that for all $c \in C$ and $\widehat{a} \in \widehat{A}$,

$$\widehat{\delta}_C\big(\delta_C(c)(1 \underset{\mathfrak{H}}{\otimes} \widehat{a})\big) = \big(\delta_C(c) \underset{\mathfrak{H}}{\otimes} 1\big)\big(1 \underset{\mathfrak{H}}{\otimes} \widehat{\Delta}(\widehat{a})\big). \tag{14}$$

If the coaction $(H, A, \widehat{\beta}, \widehat{\Delta})$ is fine, then also the coaction (13) is fine.

Proof. The triple $(K_\gamma \underset{\mathfrak{H}}{\otimes}_\beta H, C \rtimes_r \widehat{A}, \gamma \triangleright \widehat{\beta})$ is a concrete C^*-$_{\mathfrak{B}^\dagger}\mathfrak{H}_\mathfrak{B}$-algebra because

$$\rho_{(\gamma \triangleright \widehat{\beta})}(\mathfrak{B})(C \rtimes_r \widehat{A}) \subseteq \big[(1 \underset{\mathfrak{H}}{\otimes} \rho_{\widehat{\beta}}(\mathfrak{B})\widehat{A})\delta_C(C)\big] = C \rtimes_r \widehat{A}$$

by Proposition 4.5. Moreover, $C \rtimes_r \widehat{A}$ is nondegenerate because $\delta_C(C)$ and $1 \underset{\mathfrak{H}}{\otimes} \widehat{A}$ are nondegenerate (see Remark 4.2 i)).

Consider the $*$-homomorphism

$$\widehat{\delta}_C \colon C \rtimes_r \widehat{A} \to \mathcal{L}(K_\gamma \underset{\mathfrak{H}}{\otimes}_\beta H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_\alpha H), \quad x \mapsto (1 \underset{\mathfrak{H}}{\otimes} \check{V})(x \underset{\mathfrak{H}}{\otimes} 1)(1 \underset{\mathfrak{H}}{\otimes} \check{V}^*).$$

Since $\check{V}(a \underset{\mathfrak{H}}{\otimes} 1)\check{V}^* = a \underset{\mathfrak{H}}{\otimes} 1$ for all $a \in A$ (Lemma 3.5) and $\delta_C(C) \subseteq C_{\gamma} \underset{\mathfrak{H}}{*}_\beta M(A)$,

$$(1 \underset{\mathfrak{H}}{\otimes} \check{V})(\delta_C(c) \underset{\mathfrak{H}}{\otimes} 1)(1 \underset{\mathfrak{H}}{\otimes} \check{V}^*) = \delta_C(c) \underset{\mathfrak{H}}{\otimes} 1 \quad \text{for all } c \in C.$$

Moreover, by Proposition 3.3, $\check{V}(\widehat{a} \underset{\mathfrak{H}}{\otimes} 1)\check{V}^* = \widehat{\Delta}(\widehat{a})$ for all $\widehat{a} \in \widehat{A}$. Consequently, Equation (14) holds for all $c \in C$ and $\widehat{a} \in \widehat{A}$.

Let us show that $\widehat{\delta}_C(C \rtimes_r \widehat{A}) \subseteq (C \rtimes_r \widehat{A})_{\gamma \triangleright \widehat{\beta}} *_\alpha M(\widehat{A})$. By Proposition 4.5 and Equation (14),

$$\widehat{\delta}_C(C \rtimes_r \widehat{A}) = \big[\big(\delta_C(C) \underset{\mathfrak{H}}{\otimes} 1\big)\big(1 \underset{\mathfrak{H}}{\otimes} \widehat{\Delta}(\widehat{A})\big)\big].$$

13

Since $\widehat{\Delta}(\widehat{A})|\alpha\rangle_{[2]} \subseteq \big[|\alpha\rangle_{[2]}\widehat{A}\big]$,

$$\widehat{\delta}_C(C \rtimes_r \widehat{A})|\alpha\rangle_{[3]} \subseteq \big[\big(\delta_C(C)\underset{\mathfrak{H}}{\otimes}1\big)|\alpha\rangle_{[3]}\big(1\underset{\mathfrak{H}}{\otimes}\widehat{A}\big)\big]$$

$$= \big[|\alpha\rangle_{[3]}\delta_C(C)\big(1\underset{\mathfrak{H}}{\otimes}\widehat{A}\big)\big] = \big[|\alpha\rangle_{[3]}(C\rtimes_r\widehat{A})\big]. \tag{15}$$

On the other hand, since $A\widehat{\beta} \subseteq \widehat{\beta}$ [10, Lemma 4.5],

$$\delta_C(C)|\gamma\triangleright\widehat{\beta}\rangle \subseteq \big[(C_{\gamma\underset{\mathfrak{H}}{*}\beta}M(A))|\gamma\rangle_{[1]}\widehat{\beta}\big] \subseteq \big[|\gamma\rangle_{[1]}M(A)\widehat{\beta}\big] = |\gamma\rangle_{[1]}\widehat{\beta} = |\gamma\triangleright\widehat{\beta}\rangle,$$

and therefore,

$$\widehat{\delta}_C(C\rtimes_r\widehat{A})|\gamma\triangleright\widehat{\beta}\rangle_{[12]} = \big[\big(1\underset{\mathfrak{H}}{\otimes}\widehat{\Delta}(\widehat{A})\big)\big(\delta_C(C)\underset{\mathfrak{H}}{\otimes}1\big)|\gamma\triangleright\widehat{\beta}\rangle_{[12]}\big]$$

$$\subseteq \big[\big(1\underset{\mathfrak{H}}{\otimes}\widehat{\Delta}(\widehat{A})\big)|\gamma\triangleright\widehat{\beta}\rangle_{[12]}\big]$$

$$= \big[|\gamma\rangle_{[1]}\widehat{\Delta}(\widehat{A})|\widehat{\beta}\rangle_{[1]}\big] \subseteq \big[|\gamma\rangle_{[1]}|\widehat{\beta}\rangle_{[1]}\widehat{A}\big] = \big[|\gamma\triangleright\widehat{\beta}\rangle_{[12]}\widehat{A}\big].$$

Let us show that the $*$-homomorphism $\widehat{\delta}_C$ is a morphism of concrete C^*-$_{\mathfrak{B}\dagger}\mathfrak{H}_{\mathfrak{B}}$-algebras. By definition of $\widehat{\delta}_C$,

$$\widehat{\delta}_C(x)\big(1\underset{\mathfrak{H}}{\otimes}\check{V}\big)|\xi\rangle_{[3]} = \big(1\underset{\mathfrak{H}}{\otimes}\check{V}\big)\big(x\underset{\mathfrak{H}}{\otimes}1\big)|\xi\rangle_{[3]} = \big(1\underset{\mathfrak{H}}{\otimes}\check{V}\big)|\xi\rangle_{[3]}x$$

for each $x \in C\rtimes_r\widehat{A}$ and $\xi \in \widehat{\beta}$, and therefore,

$$\gamma\triangleright\widehat{\beta}\triangleright\widehat{\beta} = \gamma\triangleright\check{V}_*(\widehat{\beta}\triangleleft\widehat{\beta}) = \big[\big(1\underset{\mathfrak{H}}{\otimes}\check{V}\big)|\widehat{\beta}\rangle_{[3]}(\gamma\triangleright\widehat{\beta})\big]$$

$$\subseteq \big[\mathcal{L}^{\widehat{\delta}_C}\big((K_{\gamma\underset{\mathfrak{H}}{*}\beta}H)_{\gamma\triangleright\widehat{\beta}},(K_{\gamma\underset{\mathfrak{H}}{*}\beta}H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H)_{\gamma\triangleright\widehat{\beta}\triangleright\widehat{\beta}}\big)(\gamma\triangleright\widehat{\beta})\big].$$

Equation (14) implies that the $*$-homomorphisms

$$\big(\widehat{\delta}_C*\mathrm{id}\big)\circ\widehat{\delta}_C, \big(\mathrm{id}*\widehat{\Delta}\big)\circ\widehat{\delta}_C\colon C\rtimes_r\widehat{A}\to\mathcal{L}\big(K_\gamma\underset{\mathfrak{H}}{\otimes}_\beta H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H\big)$$

are both given by $\delta_C(c)\big(1\underset{\mathfrak{H}}{\otimes}\widehat{a}\big) \mapsto \big(\delta_C(c)\underset{\mathfrak{H}}{\otimes}1\underset{\mathfrak{H}}{\otimes}1\big)\big(1\underset{\mathfrak{H}}{\otimes}\widehat{\Delta}^{(2)}(\widehat{a})\big)$ for all $c \in C$, $\widehat{a} \in \widehat{A}$, where $\widehat{\Delta}^{(2)} := \big(\widehat{\Delta}*\mathrm{id}\big)\circ\widehat{\Delta} = \big(\mathrm{id}*\widehat{\Delta}\big)\circ\widehat{\Delta}\colon \widehat{A}\to\mathcal{L}\big(H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H\big)$.

Summarizing, we find that (13) is a coaction of $(_\alpha\widehat{A}_{\widehat{\beta}},\widehat{\Delta})$ as claimed. By construction, $\widehat{\delta}_C$ is injective, and Equation (15) shows that this coaction is fine if the coaction $(H,A,\widehat{\beta},\widehat{\Delta})$ is fine. $\square$

Definition 4.8. Let (K,C,γ,δ_C) be a coaction of $(_\beta A_\alpha,\Delta)$. We call $\big(K_\gamma\underset{\mathfrak{H}}{\otimes}_\beta H, C\rtimes_r\widehat{A}, \gamma\triangleright\widehat{\beta}, \widehat{\delta}_C\big)$ the associated dual coaction of $(_\alpha\widehat{A}_{\widehat{\beta}},\widehat{\Delta})$.

The reduced crossed product construction is functorial in the following sense:

Proposition 4.9. Let ρ be a covariant morphism between a coaction (K,C,γ,δ_C) and a coaction (L,D,δ,δ_D) of $(_\beta A_\alpha,\Delta)$. Then there exists a unique covariant morphism $\rho\rtimes_r\mathrm{id}$ between the associated dual coactions such that for all $c \in C$ and $\widehat{a} \in \widehat{A}$,

$$\big(\rho\rtimes_r\mathrm{id}\big)\big(\delta_C(c)\big(1\underset{\mathfrak{H}}{\otimes}\widehat{a}\big)\big) = \delta_D(\rho(c))\big(1\underset{\mathfrak{H}}{\otimes}\widehat{a}\big). \tag{16}$$

14

Proof. By Remark 4.4, $C \rtimes_r \widehat{A} \subseteq C_\gamma \underset{\mathfrak{H}}{*}_\beta \mathcal{L}(H)$ and $D \rtimes_r \widehat{A} \subseteq D_\delta \underset{\mathfrak{H}}{*}_\beta \mathcal{L}(H)$, and by [10, Theorem 3.15], there exists a morphism

$$\rho * \mathrm{id} \in \mathrm{Mor}\left((C_\gamma \underset{\mathfrak{H}}{*}_\beta \mathcal{L}(H))_{\gamma \triangleright \widehat{\beta}}, (D_\delta \underset{\mathfrak{H}}{*}_\beta \mathcal{L}(H))_{\delta \triangleright \widehat{\beta}}\right).$$

Denote by $\rho \rtimes_r \mathrm{id} \colon C \rtimes_r \widehat{A} \to D_\delta \underset{\mathfrak{H}}{*}_\beta \mathcal{L}(H)$ the restriction of $\rho * \mathrm{id}$. Then

$$(\rho \rtimes_r \mathrm{id})\left(\delta_C(c)(1 \underset{\mathfrak{H}}{\otimes} \widehat{a})\right) = (\rho * \mathrm{id})(\delta_C(c)) \cdot (\rho * \mathrm{id})(1 \underset{\mathfrak{H}}{\otimes} \widehat{a}) = \delta_D(\rho(c))(1 \underset{\mathfrak{H}}{\otimes} \widehat{a})$$

for all $c \in C$ and $\widehat{a} \in \widehat{A}$. In particular, $\rho \rtimes_r \mathrm{id} \in \mathrm{Mor}\left((C \rtimes_r \widehat{A})_{\gamma \triangleright \widehat{\beta}}, (D \rtimes_r \widehat{A})_{\delta \triangleright \widehat{\beta}}\right)$. Finally, Equations (14) and (16) imply that $\rho \rtimes_r \mathrm{id}$ is covariant with respect to $\widehat{\delta}_C$ and $\widehat{\delta}_D$. $\qquad\square$

Corollary 4.10. *The assignments* $(K, C, \gamma, \delta_C) \mapsto (K_\gamma \underset{\mathfrak{H}}{\otimes}_\beta H, C \rtimes_r \widehat{A}, \gamma \triangleright \widehat{\beta}, \widehat{\delta}_C)$ *and* $\rho \mapsto \rho \rtimes_r \mathrm{id}$ *define a functor from the category of coactions of* $({}_\beta A_\alpha, \Delta)$ *to the category of coactions of* $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. $\qquad\square$

Reduced crossed products for coactions of $\left({}_{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}}, H, \widehat{A}, \widehat{\beta}, \alpha, \widehat{\Delta}\right)$ We extend the reduced crossed product construction to coactions of the concrete Hopf C*-bimodule $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$ as follows:

Definition 4.11. *Let* (K, C, γ, δ_C) *be a coaction of* $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. *The associated* reduced crossed product *is the C*-subalgebra* $C \rtimes_{\delta_C, r} A \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{H}}{\otimes}_\alpha H)$ *generated by*

$$\delta(C)\left(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(A)\right) \subseteq \mathcal{L}(K_\gamma \underset{\mathfrak{H}}{\otimes}_\alpha H). \tag{17}$$

If δ_C *is understood, we shortly write* $C \rtimes_r A$ *for* $C \rtimes_{\delta_C, r} A$.

This construction has the same formal properties like the reduced crossed product for coactions of $({}_\beta A_\alpha, \Delta)$; for the proofs, one simply replaces the weak C*-pseudo-Kac system $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ by its predual, applies Proposition 3.3, and uses the results of the preceding paragraph.

Proposition 4.12. *Let* (K, C, γ, δ_C) *be a coaction of* $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. *Then* $C \rtimes_r A = \left[\delta_C(C)\left(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(A)\right)\right]$. $\qquad\square$

Theorem 4.13. *Let* (K, C, γ, δ_C) *be a coaction of* $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. *There exists a coaction* $\left(K_\gamma \underset{\mathfrak{H}}{\otimes}_\alpha H, C \rtimes_r A, \gamma \triangleright \widehat{\alpha}, \widehat{\delta}_C\right)$ *of* $({}_\beta A_\alpha, \Delta)$ *such that*

$$\widehat{\delta}_C\left(\delta_C(c)(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(a))\right) = \left(\delta_C(c) \underset{\mathfrak{H}}{\otimes} 1\right)\left(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} 1)} \Delta(a)\right)$$

for all $c \in C$ *and* $a \in A$. *If the coaction* (H, A, α, Δ) *is fine, then also* $\left(K_\gamma \underset{\mathfrak{H}}{\otimes}_\alpha H, C \rtimes_r A, \gamma \triangleright \widehat{\alpha}, \widehat{\delta}_C\right)$ *is fine.* $\qquad\square$

Definition 4.14. *Let* (K, C, γ, δ_C) *be a coaction of* $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. *Then we call* $\left(K_\gamma \underset{\mathfrak{H}}{\otimes}_\alpha H, C \rtimes_r A, \gamma \triangleright \widehat{\alpha}, \widehat{\delta}_C\right)$ *the associated* dual coaction *of* $({}_\beta A_\alpha, \Delta)$.

Proposition 4.15. *Let* ρ *be a covariant morphism between a coaction* (K, C, γ, δ_C) *and a coaction* (L, D, δ, δ_D) *of* $({}_\alpha \widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. *Then there exists a unique covariant morphism* $\rho \rtimes_r \mathrm{id}$ *between the associated dual coactions such that*

$$(\rho \rtimes_r \mathrm{id})\left(\delta_C(c)(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(a))\right) = \delta_D(\rho(c))(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(a))$$

for all $c \in C$ *and* $a \in A$. $\qquad\square$

15

Corollary 4.16. *The assignments* $(K, C, \gamma, \delta_C) \mapsto \left(K_{\gamma} \underset{\mathfrak{H}}{\otimes}_{\alpha} H, C \rtimes_r A, \gamma \triangleright \widehat{\alpha}, \widehat{\delta_C}\right)$ *and* $\rho \mapsto$ $\rho \rtimes_r \mathrm{id}$ *define a functor from the category of coactions of* $(_{\alpha}\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$ *to the category of coactions of* $(_{\beta}A_{\alpha}, \Delta)$. $\qquad\square$

5 C^*-pseudo-Kac systems

To obtain an analogue of Baaj-Skandalis duality [1], we need to refine the notion of a weak C^*-pseudo-Kac system as follows.

As before, we fix the C^*-base $_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}$ and the Hilbert space H.

Definition 5.1. *We call a balanced C^*-pseudo-multiplicative unitary* $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *a C^*-pseudo-Kac-system if*

  i) the C^*-pseudo-multiplicative unitaries $V, \check{V}, \widehat{V}$ are regular, and

  ii) $\left(\Sigma(1 \underset{\mathfrak{H}}{\otimes} U)V\right)^3 = 1$ *in* $\mathcal{L}(H_{\widehat{\beta}} \underset{\mathfrak{H}}{\otimes}_{\alpha} H)$.

Remark 5.2. In leg notation, the equation $\left(\Sigma(1 \underset{\mathfrak{H}}{\otimes} U)V\right)^3 = 1$ takes the form $(\Sigma U_{[2]} V)^3 = 1$. Conjugating by Σ or V, we find that this condition is equivalent to the relation $(U_{[2]} V \Sigma)^3 = 1$ and to the relation $(V \Sigma U_{[2]})^3 = 1$.

Remark 5.3. Definition 5.1 greatly simplifies the definition of a pseudo-Kac system on C^*-modules given in [9] and still covers our main examples. The two concepts are related as follows.

Let $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ be a C^*-pseudo-Kac system and assume that $\mathfrak{B}^\dagger \cong \mathfrak{B}^{op}$. Then the C^*-modules $\alpha, \widehat{\alpha}, \beta, \widehat{\beta}$, the representations ρ_{α}, $\rho_{\widehat{\alpha}}$, ρ_{β}, $\rho_{\widehat{\beta}}$, the unitaries $U_{\alpha} \colon \alpha \to \widehat{\alpha}$, $U_{\widehat{\alpha}} \colon \widehat{\alpha} \to \alpha$, $U_{\beta} \colon \beta \to \widehat{\beta}$, $U_{\widehat{\beta}} \colon \widehat{\beta} \to \beta$ together with the family of unitaries

$$V_{\alpha \triangleleft \alpha} \colon \alpha_{\rho_{\widehat{\beta}}} \otimes \alpha \to \alpha \otimes_{\rho_{\beta}} \alpha, \qquad V_{\widehat{\beta} \triangleright \beta} \colon \widehat{\beta} \otimes_{\rho_{\alpha}} \beta \to \widehat{\beta}_{\rho_{\alpha}} \otimes \beta, \qquad V_{\widehat{\alpha} \triangleleft \alpha} \colon \widehat{\alpha}_{\rho_{\widehat{\beta}}} \otimes \alpha \to \widehat{\alpha}_{\rho_{\alpha}} \otimes \beta,$$

$$V_{\widehat{\beta} \triangleright \widehat{\beta}} \colon \widehat{\beta} \otimes_{\rho_{\alpha}} \widehat{\beta} \to \alpha \otimes_{\rho_{\beta}} \widehat{\beta}, \qquad V_{\beta \triangleleft \alpha} \colon \beta_{\rho_{\widehat{\beta}}} \otimes \alpha \to \beta_{\rho_{\alpha}} \otimes \beta, \qquad V_{\widehat{\beta} \triangleright \widehat{\alpha}} \colon \widehat{\beta} \otimes_{\rho_{\alpha}} \widehat{\alpha} \to \alpha \otimes_{\rho_{\beta}} \widehat{\alpha}$$

form a pseudo-Kac system in the sense of [9, Definition 2.53].

We shall use the following reformulation of condition ii) in Definition 5.1:

Lemma 5.4. *Let* $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *be a balanced C^*-pseudo-multiplicative unitary. Then* $(\Sigma U_{[2]} V)^3 = 1$ *if and only if* $\widehat{V} V \check{V} = U_{[1]} \Sigma$.

Proof. Rearranging the factors in the product $U_{[1]} U_{[2]} \left(\Sigma U_{[2]} V\right)^3 U_{[2]} \Sigma$, we find

$$U_{[1]} U_{[2]} (\Sigma U_{[2]} V)^3 U_{[2]} \Sigma = \Sigma U_{[1]} V U_{[1]} \Sigma \cdot V \cdot \Sigma U_{[2]} V U_{[2]} \Sigma = \widehat{V} \cdot V \cdot \check{V}.$$

Thus, $(\Sigma U_{[2]} V)^3$ is equal to 1 if and only if $U_{[1]} \Sigma = U_{[1]} U_{[2]} U_{[2]} \Sigma$ is equal to $\widehat{V} V \check{V}$. $\qquad\square$

The notion of a C^*-pseudo-Kac system is symmetric in the following sense:

Proposition 5.5. *Let* $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *be a C^*-pseudo-Kac system. Then also the tuples* (11) *are C^*-pseudo-Kac systems.*

Proof. Equations (6), (7) and [10, Remark 4.11] imply that the tuples (11) satisfy condition i) of Definition 5.1. To check that they also satisfy condition ii), we use Remark 5.2:

$$(\Sigma U_{[2]} \widehat{V})^3 = (V \Sigma U_{[2]})^3 = 1, \qquad (\check{V} \Sigma U_{[2]})^3 = (\Sigma U_{[2]} V)^3 = 1,$$

$$(U_2 V^{op} \Sigma)^3 = (U_{[2]} \Sigma V^*)^3 = \left((V \Sigma U_{[2]})^3\right)^* = 1. \qquad\square$$

The following two propositions are essential for our duality theorem:

Proposition 5.6. *Let* $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *be a C^*-pseudo-Kac system. Then it is a weak C^*-pseudo-Kac system.*

Proof. Since V is regular, it is well-behaved. We show that condition i)(a) in Lemma 3.5 holds. By Lemma 3.4, $\check{V}_{[12]}V_{[13]} = V_{[13]}V_{[23]}\check{V}_{[12]}$. We multiply this equation by $V_{[12]}$ on the left and by $\Sigma_{[12]}$ on the right, use the pentagon equation for V, and obtain

$$V_{[12]}\check{V}_{[12]}\Sigma_{[12]}V_{[23]} = V_{[12]}\check{V}_{[12]}V_{[13]}\Sigma_{[12]} = V_{[12]}V_{[13]}V_{[23]}\check{V}_{[12]}\Sigma_{[12]} = V_{[23]}V_{[12]}\check{V}_{[12]}\Sigma_{[12]}.$$

By Lemma 5.4, we can replace $V_{[12]}\check{V}_{[12]}\Sigma_{[12]}$ by $\widehat{V}_{[12]}^{*}U_{[1]}$, and find that $\widehat{V}_{[12]}^{*}U_{[1]}V_{[23]} = V_{[23]}\widehat{V}_{[12]}^{*}U_{[1]}$. Since $\widehat{V}_{[12]}$ is unitary and $U_{[1]}V_{[23]} = V_{[23]}U_{[1]}$, we can conclude $\widehat{V}_{[12]}V_{[23]} = V_{[23]}\widehat{V}_{[12]}$, that is, condition i)(a) in Lemma 3.5 holds. A similar argument shows that condition ii)(a) in Lemma 3.5 holds. $\square$

Proposition 5.7. *Let* $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ *be a C^*-pseudo-Kac system. Then* $\big[A(V)\widehat{A}(V)\big] = \big[\widehat{\alpha}\widehat{\alpha}^{*}\big]$.

We need the following lemma:

Lemma 5.8. *Let $V\colon H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_{\alpha}H \to H_{\alpha}\underset{\mathfrak{H}}{\otimes}_{\beta}H$ be a regular C^*-pseudo-multiplicative unitary. Then* $\big[V|\alpha\rangle_{[2]}\widehat{A}(V)\big] = \big[|\beta\rangle_{[2]}\widehat{A}(V)\big]$.

Proof. The first commutative diagram in [10, Proof of Proposition 4.12] shows

$$\big[V|\alpha\rangle_{[2]}\widehat{A}(V)\big] = \big[V|\alpha\rangle_{[2]}\widehat{A}(V)^{*}\big] = \big[\langle\alpha|_{[2]}V_{[12]}^{*}|\beta\rangle_{[3]}|\beta\rangle_{[2]}\big],$$

and the following commutative diagram shows that this equals $\big[|\beta\rangle_{[2]}\widehat{A}(V)^{*}\big] = \big[|\beta\rangle_{[2]}\widehat{A}(V)\big]$:

Proof of Proposition 5.7. Since V is regular and $V^{*} = \Sigma U_{[2]}V\Sigma U_{[2]}V\Sigma U_{[2]}$,

$$\big[\widehat{\alpha}\widehat{\alpha}^{*}\big] = \big[U\alpha\alpha^{*}U\big] = \big[U\langle\alpha|_{[2]}V^{*}|\alpha\rangle_{[1]}U\big]$$
$$= \big[U\langle\alpha|_{[2]}\Sigma U_{[2]}V\Sigma U_{[2]}V\Sigma U_{[2]}|\alpha\rangle_{[1]}U\big] = \big[\langle\alpha|_{[1]}V\Sigma U_{[2]}V|\alpha\rangle_{[2]}\big].$$

We multiply on the right by $\widehat{A} := \widehat{A}(V)$, use Equation (8) and Lemma 5.8, and find

$$\big[\widehat{\alpha}\widehat{\alpha}^{*}\big] = \big[\widehat{\alpha}\widehat{\alpha}^{*}\widehat{A}\big] = \big[\langle\alpha|_{[1]}V\Sigma U_{[2]}V|\alpha\rangle_{[2]}\widehat{A}\big]$$
$$= \big[\langle\alpha|_{[1]}V\Sigma U_{[2]}|\beta\rangle_{[2]}\widehat{A}\big] = \big[\langle\alpha|_{[1]}V|\widehat{\beta}\rangle_{[1]}\widehat{A}\big] = \big[A(V)\widehat{A}(V)\big]. \quad \square$$

The duality theorem Let $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ be a C^*-pseudo-Kac system, put

$$\widehat{A} := \widehat{A}(V), \qquad \widehat{\Delta} := \widehat{\Delta}_V, \qquad A := A(V), \qquad \Delta := \Delta_V,$$

and write $({}_\alpha\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$ and $({}_\beta A_\alpha, \Delta)$ for the concrete Hopf C^*-bimodules $\big({}_{\mathfrak{B}^\dagger}\mathfrak{H}_{\mathfrak{B}}, H, \widehat{A}, \widehat{\beta}, \alpha, \widehat{\Delta}\big)$ and $\big({}_{\mathfrak{B}}\mathfrak{H}_{\mathfrak{B}^\dagger}, H, A, \alpha, \beta, \Delta\big)$, respectively.

Before we can state the duality theorem, we need some preliminaries. Let (K, C, γ, δ_C) be a coaction of $({}_\beta A_\alpha, \Delta)$. By Lemma 2.6, $\big[|\beta\rangle_{[2]} C \langle \beta|_{[2]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H)$ is a C^*-algebra and there exists a $*$-homomorphism

$$\mathrm{Ind}_\beta(\delta_C) \colon \big[|\beta\rangle_{[2]} C \langle \beta|_{[2]}\big] \to \big[|\beta\rangle_{[3]} \delta_C(C) \langle \beta|_{[3]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H {}_\alpha\underset{\mathfrak{H}}{\otimes}{}_\beta H).$$

Likewise, if (K, C, γ, δ_C) is a coaction of $({}_\alpha\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$, then $\big[|\alpha\rangle_{[2]} C \langle \alpha|_{[2]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\alpha H)$ is a C^*-algebra and there exists a $*$-homomorphism

$$\mathrm{Ind}_\alpha(\delta_C) \colon \big[|\alpha\rangle_{[2]} C \langle \alpha|_{[2]}\big] \to \big[|\alpha\rangle_{[3]} \delta_C(C) \langle \alpha|_{[3]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\alpha H {}_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}{}_\alpha H).$$

Theorem 5.9.   *i) Let (K, C, γ, δ_C) be a fine coaction of $({}_\beta A_\alpha, \Delta)$. There exists a natural isomorphism*

$$C \rtimes_r \widehat{A} \rtimes_r A \cong \big[|\beta\rangle_{[2]} C \langle \beta|_{[2]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H)$$

which identifies the bidual coaction $\widehat{\widehat{\delta}}_C$ on $C \rtimes_r \widehat{A} \rtimes_r A$ with the map $\mathrm{Ad}_{(1 \underset{\mathfrak{H}}{\otimes} \Sigma \widehat{V})} \circ \mathrm{Ind}_\beta(\delta_C)$.

ii) Let (K, C, γ, δ_C) be a fine coaction of $({}_\alpha\widehat{A}_{\widehat{\beta}}, \widehat{\Delta})$. There exists a natural isomorphism

$$C \rtimes_r A \rtimes_r \widehat{A} \cong \big[|\alpha\rangle_{[2]} C \langle \alpha|_{[2]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\alpha H)$$

which identifies the bidual coaction $\widehat{\widehat{\delta}}_C$ on $C \rtimes_r A \rtimes_r \widehat{A}$ with the map $\mathrm{Ad}_{(1 \underset{\mathfrak{H}}{\otimes} \Sigma V)} \circ \mathrm{Ind}_\alpha(\delta_C)$.

Proof. i) By Proposition 4.12, Definition 4.8, and Proposition 4.5, the iterated crossed product $C \rtimes_r \widehat{A} \rtimes_r A$ is equal to

$$\Big[\big(\delta_C(C) \underset{\mathfrak{H}}{\otimes} 1\big)\big(1 \underset{\mathfrak{H}}{\otimes} \widehat{\Delta}(\widehat{A})\big)\big(1 \underset{\mathfrak{H}}{\otimes} 1 \otimes \mathrm{Ad}_U(A)\big)\Big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H {}_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}{}_\alpha H).$$

By definition of Δ and $\widehat{\Delta}$ and by Lemma 3.5, conjugation by $1 \underset{\mathfrak{H}}{\otimes} V$ maps this C^*-algebra isomorphically onto

$$\Big[\delta_C^{(2)}(C)\big(1 \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} \widehat{A}\big)\big(1 \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(A)\big)\Big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H {}_\alpha\underset{\mathfrak{H}}{\otimes}{}_\beta H),$$

where $\delta_C^{(2)} = (\mathrm{id} * \Delta) \circ \delta_C = (\delta_C * \mathrm{id}) \circ \delta_C$. Since δ_C is injective, the formula

$$\delta_C^{(2)}(c)(1 \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} T) \mapsto \delta_C(c)(1 \underset{\mathfrak{H}}{\otimes} T), \quad \text{where } c \in C, \ T \in \mathcal{L}(H),$$

defines an isomorphism of this C^*-algebra onto

$$\Big[\delta_C(C)\big(1 \underset{\mathfrak{H}}{\otimes} \widehat{A} \cdot \mathrm{Ad}_U(A)\big)\Big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H). \tag{18}$$

By Proposition 3.3 and Proposition 5.7, applied to the C^*-pseudo-Kac system $(\widehat{\beta}, \beta, \alpha, \widehat{\alpha}, U, \check{V})$, we have $\big[\widehat{A} \, \mathrm{Ad}_U(A)\big] = \big[A(\check{V})\widehat{A}(\check{V})\big] = \big[\beta\beta^*\big]$. We insert this relation into (18), use the fact that δ_C is a fine coaction, and find

$$C \rtimes_r \widehat{A} \rtimes_r A \cong \big[\delta(C)|\beta\rangle_{[2]} \langle \beta|_{[2]}\big] = \big[|\beta\rangle_{[2]} C \langle \beta|_{[2]}\big] \subseteq \mathcal{L}(K_{\gamma} \underset{\mathfrak{H}}{\otimes}{}_\beta H).$$

18

This isomorphism identifies an element of the form

$$\left(\delta_C(c) \underset{\mathfrak{H}}{\otimes} 1\right)\left(1 \underset{\mathfrak{H}}{\otimes} \widehat{\Delta}(\hat{a})\right)\left(1 \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(a)\right) \in C \rtimes_r \widehat{A} \rtimes_r A \tag{19}$$

with

$$\delta_C(c)\left(1 \underset{\mathfrak{H}}{\otimes} \hat{a} \cdot \mathrm{Ad}_U(a)\right) \in \left[|\beta\rangle_{[2]} C\langle\beta|_{[2]}\right]. \tag{20}$$

The bidual coaction $\widehat{\widehat{\delta}}_C$ maps the element (19) to

$$\left(\delta_C(c) \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} 1\right)\left(1 \underset{\mathfrak{H}}{\otimes} \widehat{\Delta}(\hat{a}) \underset{\mathfrak{H}}{\otimes} 1\right)\left(1 \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} 1)}(\Delta(a))\right),$$

and the map $\mathrm{Ad}_{(1 \underset{\mathfrak{H}}{\otimes} \Sigma \widehat{V})} \circ \mathrm{Ind}_\beta(\delta_C)$ sends the element (20) to

$$\mathrm{Ad}_{(1 \underset{\mathfrak{H}}{\otimes} \Sigma \widehat{V})}\left(\delta_C^{(2)}(c)(1 \underset{\mathfrak{H}}{\otimes} 1 \underset{\mathfrak{H}}{\otimes} \hat{a} \cdot \mathrm{Ad}_U(a))\right) = \left(\delta_C(c) \underset{\mathfrak{H}}{\otimes} 1\right)\left(1 \underset{\mathfrak{H}}{\otimes} \hat{a} \underset{\mathfrak{H}}{\otimes} 1\right)\left(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} 1)}(\Delta(a))\right);$$

here, we used the following relations:

- $\mathrm{Ad}_{\Sigma\widehat{V}}(\Delta(b)) = \mathrm{Ad}_\Sigma(1 \underset{\mathfrak{H}}{\otimes} b) = b \underset{\mathfrak{H}}{\otimes} 1$ for all $b \in A$ by Proposition 3.3;

- $\mathrm{Ad}_{\Sigma\widehat{V}}(1 \underset{\mathfrak{H}}{\otimes} \hat{a}) = \mathrm{Ad}_\Sigma(1 \underset{\mathfrak{H}}{\otimes} \hat{a}) = \hat{a} \underset{\mathfrak{H}}{\otimes} 1$ by Lemma 3.5;

- $\mathrm{Ad}_{\Sigma\widehat{V}}(1 \underset{\mathfrak{H}}{\otimes} \mathrm{Ad}_U(a)) = \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} 1)}\left(\mathrm{Ad}_V(1 \underset{\mathfrak{H}}{\otimes} a)\right) = \mathrm{Ad}_{(U \underset{\mathfrak{H}}{\otimes} 1)}(\Delta(a))$ by definition of Δ.

Therefore, the bidual coaction $\widehat{\widehat{\delta}}_C$ on $C \rtimes_r \widehat{A} \rtimes_r A$ corresponds to the map $\mathrm{Ad}_{(1 \underset{\mathfrak{H}}{\otimes} \Sigma\widehat{V})} \circ \mathrm{Ind}_\beta(\delta_C)$.

ii) The proof is similar to the proof of i), simply replace the C^*-pseudo-Kac system $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, U, V)$ by its predual $(\widehat{\beta}, \beta, \alpha, \widehat{\alpha}, U, \check{V})$. $\qquad\square$

6   The C^*-pseudo-Kac system of a locally compact groupoid

The prototypical example of a C^*-pseudo-Kac system is the one associated to a locally compact groupoid. The underlying C^*-pseudo-multiplicative unitary was described in [10]. For background on groupoids, Haar measures, and quasi-invariant measures, see [7] or [6].

The data   Let G be a locally compact, Hausdorff, second countable groupoid. We denote its unit space by G^0, its range map by r_G, its source map by s_G, and put $G^u := r_G^{-1}(\{u\})$, $G_u := s_G^{-1}(u)$ for each $u \in G^0$.

We assume that G has a left Haar system λ, and denote the associated right Haar system by λ^{-1}. Let μ be a measure on G^0 and denote by ν the measure on G given by

$$\int_G f \, d\nu := \int_{G^0} \int_{G^u} f(x) \, d\lambda^u(x) \, d\mu(u) \quad \text{for all } f \in C_c(G).$$

The push-forward of ν via the inversion map $G \to G$, $x \mapsto x^{-1}$, is denoted by ν^{-1}; evidently,

$$\int_G f \, d\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \, d\lambda_u^{-1}(x) \, d\mu(u).$$

We assume that the measure μ is quasi-invariant, i.e., that ν and ν^{-1} are equivalent. Note that there always exist sufficiently many quasi-invariant measures [7]. We denote by $D := d\nu/d\nu^{-1}$ the Radon-Nikodym derivative.

19

The C^*-base and the Hilbert space   The measure μ defines a tracial proper weight on the C^*-algebra $C_0(G^0)$, which we denote by μ again. Moreover, we denote by $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$ the C^*-base associated to μ (see Section 2); thus, $\mathfrak{H} = L^2(G^0, \mu)$. Note that $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger} = {}_{\mathfrak{B}^\dagger}\mathfrak{H}_\mathfrak{B}$ because $C_0(G^0)$ is commutative. Put $H := L^2(G, \nu)$.

The C^*-factorizations   The space $C_c(G)$ forms a pre-C^*-module over $C_0(G^0)$ with respect to the structure maps

$$\langle \xi' | \xi \rangle(u) = \int_{G^u} \overline{\xi'(x)} \xi(x) d\lambda^u(x), \quad (\xi f)(x) = \xi(x) f(r_G(x)),$$

and also with respect to the structure maps

$$\langle \xi' | \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) d\lambda_u^{-1}(x), \quad (\xi f)(x) = \xi(x) f(s_G(x)).$$

Denote the completions of these pre-C^*-modules by $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$, respectively. By [10, Proposition 5.1], there exist isometric embeddings

$$j \colon L^2(G, \lambda) \to \mathcal{L}(\mathfrak{H}, H) \qquad \text{and} \qquad \hat{j} \colon L^2(G, \lambda^{-1}) \to \mathcal{L}(\mathfrak{H}, H)$$

such that for all $\xi \in C_c(G)$, $\zeta \in L^2(G^0, \mu)$, $x \in G$,

$$\big(j(\xi)\zeta\big)(x) = \xi(x)\zeta(r_G(x)), \qquad \big(\hat{j}(\xi)\zeta\big)(x) = \xi(x) D^{-1/2}(x)\zeta(s_G(x)).$$

Moreover, by [10, Proposition 5.1], the images

$$\alpha := \beta := j(L^2(G, \lambda)) \quad \text{and} \quad \widehat{\alpha} := \widehat{\beta} := \hat{j}(L^2(G, \lambda^{-1}))$$

are compatible C^*-factorizations of H with respect to $_\mathfrak{B}\mathfrak{H}_{\mathfrak{B}^\dagger}$, the maps j and $\hat{j}$ are unitary as maps of C^*-modules over $C_0(G^0) \cong \mathfrak{B}$, and for all $x \in G$, $\xi \in C_c(G)$, and $f \in C_0(G^0)$,

$$\big(\rho_\alpha(f)\xi\big)(x) := f\big(r_G(x)\big)\xi(x), \qquad \big(\rho_{\widehat{\alpha}}(f)\xi\big)(x) := f\big(s_G(x)\big)\xi(x)$$

The C^*-pseudo-multiplicative unitary   By [12] and [10, Proposition 2.14], the Hilbert spaces $H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H$ and $H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H$ can be described as follows. Define a measure $\nu_{s,r}^2$ on $G_{s,r}^2 := \{(x,y) \in G \times G \mid s(x) = r(y)\}$ by

$$\int_{G_{s,r}^2} f \, d\nu_{s,r}^2 := \int_{G^0} \int_{G^u} \int_{G^{s_G(x)}} f(x,y) \, d\lambda^{s_G(x)}(y) \, d\lambda^u(x) \, d\mu(u),$$

and a measure $\nu_{r,r}^2$ on $G_{r,r}^2 := \{(x,y) \in G^2 \mid r_G(x) = r_G(y)\}$ by

$$\int_{G_{r,r}^2} g \, d\nu_{r,r}^2 := \int_{G^0} \int_{G^u} \int_{G^u} g(x,y) \, d\lambda^u(y) \, d\lambda^u(x) \, d\mu(u),$$

where $f \in C_c(G_{s,r}^2)$ and $g \in C_c(G_{r,r}^2)$. By [12] and [10, Theorem 5.2], there exist isomorphisms

$$H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H \cong L^2\big(G_{s,r}^2, \nu_{s,r}^2\big), \qquad H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H \cong L^2\big(G_{r,r}^2, \nu_{r,r}^2\big)$$

and a C^*-pseudo-multiplicative unitary $V \colon H_{\widehat{\beta}}\underset{\mathfrak{H}}{\otimes}_\alpha H \to H_\alpha \underset{\mathfrak{H}}{\otimes}_\beta H$ such that, with respect to these isomorphisms, $(V\zeta)(x,y) = \zeta(x, x^{-1}y)$ for all $\zeta \in L^2(G_{s,r}^2, \nu_{s,r}^2)$ and $(x,y) \in G_{r,r}^2$.

The C^*-pseudo-Kac system By definition of the Radon-Nikodym derivative $D = d\nu/d\nu^{-1}$, there exists a unitary $U \in \mathcal{L}(H)$ such that

$$(U\xi)(x) := \xi(x^{-1})D(x)^{-1/2} \quad \text{for all } x \in G, \xi \in C_c(G).$$

This unitary is a symmetry because for all $\xi \in C_c(G)$ and ν-almost all $x \in G$,

$$(U^2\xi)(x) = (U\xi)(x^{-1})D(x)^{-1/2} = \xi(x)D(x)^{1/2}D(x)^{-1/2} = \xi(x);$$

here, we used the relation $D(x)^{-1} = D(x^{-1})$ [7, p. 23], [6, Eq. (3.7)].

Theorem 6.1. $(\alpha, \widehat{\alpha}, \beta, \widehat{\beta}, V, U)$ *is a C^*-pseudo-Kac system.*

Proof. By [10, Proposition 5.3], the C^*-pseudo-multiplicative unitary V is regular.

We claim that $\widehat{V} = \Sigma U_{[1]} V U_{[1]} \Sigma$ is equal to $V^{op} = \Sigma V^* \Sigma$. Indeed, for all $\zeta \in L^2(G_{r,r}^2, \nu_{r,r}^2)$ and $(x,y) \in G_{s,r}^2$,

$$\begin{aligned}
(U_{[1]}VU_{[1]}\zeta)(x,y) &= (VU_{[1]}\zeta)(x^{-1},y)D(x)^{-1/2} \\
&= (U_{[1]}\zeta)(x^{-1},xy)D(x)^{-1/2} = \zeta(x,xy)D(x^{-1})^{-1/2}D(x)^{-1/2}.
\end{aligned}$$

Since $D(x^{-1})^{-1/2}D(x)^{-1/2} = 1$ for ν-almost all $x \in G$ [7, p. 23], [6, Eq. (3.7)], we can conclude $U_{[1]}VU_{[1]} = V^*$, and the claim follows.

In particular, $\widehat{V} = V^{op}$ is a regular C^*-pseudo-multiplicative unitary [10, Remark 4.11].

To finish the proof, we only need to show that the map $Z := \Sigma U_{[2]} V \colon H \, {}_{\widehat{\beta}}\!\underset{\mathfrak{H}}{\otimes}{}_\alpha H \to H \, {}_{\widehat{\beta}}\!\underset{\mathfrak{H}}{\otimes}{}_\alpha H$ satisfies $Z^3 = 1$. For all $\zeta \in L^2(G_{s,r}^2, \nu_{s,r}^2)$ and $(x,y) \in G_{s,r}^2$,

$$(\Sigma U_{[2]}V\zeta)(x,y) = (V\zeta)(y,x^{-1})D(x)^{-1/2} = \zeta(y,y^{-1}x^{-1})D(x)^{-1/2},$$

and therefore,

$$\begin{aligned}
(Z^3\zeta)(x,y) &= (Z^2\zeta)(y,y^{-1}x^{-1})D(x)^{-1/2} \\
&= (Z\zeta)(y^{-1}x^{-1},xyy^{-1})\big(D(x)D(y)\big)^{-1/2} \\
&= \zeta(x,x^{-1}xy)\big(D(x)D(y)D(y^{-1}x^{-1})\big)^{-1/2}.
\end{aligned}$$

Since $D(x)D(y)D(y^{-1}x^{-1}) = 1$ for $\nu_{s,r}^2$-almost all $(x,y) \in G_{s,r}^2$ [3] (but see also [6, p. 89]), we can conclude $Z = \mathrm{id}$. $\qquad\square$

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